On the derivation of non-local diffusion equations in confined spaces

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Thesis Summary

ON THE DERIVATION OF NON-LOCAL DIFFUSION EQUATIONS IN CONFINED SPACES

Nonlocal diffusion equations are partial differential equations that model the fractional diffusion phenomena observed, for instance, in plasma physics, and have received a lot of attention in recent years. They involve fractional integro-differential operators, such as the **fractional Laplacian**. Unlike classical derivatives, these are nonlocal in the sense that the fractional derivative of a function at a point x will be influenced by the behaviour of the function in the whole domain, even far away from x. The purpose of this thesis is to understand how these nonlocal diffusion operators interact with an external electric field or with spatial boundaries. To that end, we will adopt a kinetic point of view on the diffusion process in order to have a more detailed understanding of the phenomenon, and derive from kinetic equations with geometric constraints the confined nonlocal diffusion equations.

The **fractional Vlasov-Fokker-Planck equations** are particularly adapted to this purpose. Indeed, they already feature a fractional Laplacian but it acts solely on the velocity of particles so it does not interact directly, at the kinetic scale, with the spatial confinements we introduce. We present in this thesis a method we developed to investigate the **anomalous diffusion limit** of these equations in such a way that we can track the interaction as it arises through this limit in order to construct natural macroscopic operators that are both non-local and adapted to the confinements we consider.

We will first study the fractional Vlasov-Fokker-Planck equation set on the whole space with an **external electric field** and show that its anomalous diffusion limit is an **advection-fractional diffusion equation** if the field satisfies a precise scaling property.

Then, we will set the kinetic equation in a **bounded spatial domain** and consider, on the boundary of that domain, either **absorption**, **specular reflection** or **diffusive** boundary conditions. We will investigate how each of these boundary conditions affects the diffusion inside the domain in order to construct a **new non-local diffusion operator adapted to the boundary condition**. Finally, we will established fundamental properties of these new operators and prove the well-posedness of the associated nonlocal diffusion equations.

Ludovic Cesbron

À mes parents
Lydia et Louis-Marie

Statement of Originality

I hereby declare that my dissertation entitled *On the derivation of non-local diffusion equations in confined spaces* is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is concurrently submitted for any such degree of diploma or other qualification. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

Chapter I motivates the research problems that we address in the following chapters. It gives a historical review of how our understanding of diffusion phenomena evolved since the XIXth century, presents the mathematical framework of non-local diffusion equations and introduces the challenges we face today with the confinement of non-local diffusion processes. It is my own review, based on a number of references cited throughout the chapter.

Chapter II is original research produced in collaboration with Dr. Pedro Aceves-Sánchez. It concerns the derivation of non-local advection-diffusion equations with an external electric field. This research problem was suggested by Prof. Christian Schmeiser.

Chapter III is original work, it is the core of this thesis and addresses the original question around which my Ph.D. is articulated, which is the derivation of non-local diffusion equations in bounded domain. This research problem was suggested by Prof. Antoine Mellet as a continuation of a previous collaboration ([CMT12]), and the work we present was done under the supervision and with the guidance and my Ph.D. advisors Prof. Antoine Mellet and Prof. Clément Mouhot.

Chapter IV is original work produced in collaboration with Dr. Harsha Hutridurga. It concerns the application of the method presented in Chapter III to the non-fractional case, i.e. to classical diffusion equation. The research problem arose from a discussion between Dr. Hutridurga and myself.

Chapter V presents original and unpublished results from an on-going work in collaboration with Prof. Antoine Mellet and Prof. Marjolaine Puel. It concerns the derivation of non-local diffusion equations in bounded domain from kinetic equations with diffusive boundary conditions. Note that the method we develop in this chapter is still partly formal and this work is not meant to be published individually in its current state.

Appendix A is original work, it combines the appendices of [Ces16] and [CH16] on which Chapter III and Chapter IV are based. It concerns the regularity of solutions of the free transport equation in a ball with specular reflections on the boundary. Although the results we present are tailor-made for the purposes of Chapter III and Chapter IV, we present them separately because we feel they constitute interesting results on their own and, moreover, because we adopted a Lagrangian approach to this problem and consequently the proofs are rather technical and computational.

Ludovic Cesbron
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Now that I come to the end of my Ph.D., I realise that although this thesis embodies the conclusion of these past four years, it tells little of the journey it has been. Journey of many faces, some marvellous, some wonderfully traditional in a way only Cambridge can be, and some less pleasant, sometimes even painful. I can honestly say that I could not have emerged from these challenging years without the guidance, the friendship, the support and the love of the people around me and I am delighted to have an opportunity, with these acknowledgements, to express my sincerest gratitude and thank them all.

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Chapter I

Introduction

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"La chaleur pénètre, comme la gravité, toutes les substances de l'univers, ses rayons occupent toutes les parties de l'espace. Le but de notre ouvrage est d'exposer les lois mathématiques que suit cet élément. Cette théorie formera désormais une des branches les plus importantes de la physique générale."
– *Joseph Fourier, 1822,* Théorie Analytique de la Chaleur

The term *diffusion* means "to spread out", it is the movement of a quantity (mass, heat, energy...) from a region of high concentration to regions of lower concentration. Diffusion phenomena are omnipresent in our everyday life and illustrate, in a rather explicit manner, the tendency of any natural system to evolve toward a state of equilibrium as expressed by the second law of thermodynamics which identifies this equilibrium as the state of maximum entropy. It is therefore not surprising that understanding the mathematical laws underlying diffusion phenomena has been one of the most fundamental and influential problems in the history of science as predicted by Fourier in 1822 in the quote above translated here:

> *Heat, like gravity, penetrates every substance of the universe, its rays occupy all parts of space. The object of our work is to set forth the mathematical laws which this element obeys. The theory of heat will hereafter form one of the most important branches of general physics.*

(translation by Alexander Freeman, 1878)

The purpose of this introduction is to present the crucial concepts and tools that were developed throughout history to understand and model diffusion phenomena, and the challenges we face today to model the peculiar diffusion processes observed in plasmas and turbulent fluids, especially when they are confined by an external force or in a bounded domain. The governing principle which motivates our entire presentation is the derivation of diffusion equations from "simple" mechanisms, and especially from models of motion for microscopic particles.

The first part of our introduction is devoted to classical diffusion equations which model for instance the diffusion of heat through a medium or the collective motion of particles in a rarefied gas or a fluid, near equilibrium. This is the context in which most of the key concepts in our understanding of diffusion phenomena were introduced. We first present Fourier's original derivation of the heat equation via the characterisation of the heat flux through a surface with the celebrated Fourier law. We then adopt a microscopic point of view and introduce the Brownian motion and the Langevin equation to model the motion of particles in a fluid and show how to recover the heat equation from these microscopic models. This leads us naturally to the kinetic theory of gases and the mesoscopic description of a cloud of particles. We conclude with the diffusion limit of those kinetic equations through which we recover both the Fourier law and the heat equation.

In the second part of this introduction we discuss the non-local nature of the transport and diffusion processes observed in plasmas and turbulent fluids, and the resulting generalisation of the concepts introduced in the classical setting. In order to derive the kinetic and macroscopic equations that model such phenomena, we start with a microscopic point a view and see how we can generalise the Brownian motion into Lévy flights that fit the anomalous behaviour of particles in turbulent fluids. We then introduce the fractional heat equation as well as fractional kinetic models, in particular the fractional Vlasov-Fokker-Planck equation, and conclude with the anomalous diffusion limit of the kinetic equations through which we can recover non-local diffusion equations, highlighting the compatibility of these descriptions.

Finally, in the third part, we confine the diffusion processes, either with a "soft" confinement via an external electric field, or with a "hard" confinement, restricting the process to a bounded domain. After a brief overview of the confinement of classical diffusion models we introduce the challenges that arise when we try to confine a non-local phenomena to a bounded domain, both from a microscopic and a macroscopic point of view, and present some of the most recent results on the subject.

The object of Chapter II, Chapter III and Chapter V of this thesis is the derivation of non-local diffusion equations in confined spaces from kinetic equations with a strong non-local feature. More precisely, we will mainly focus on the fractional Vlasov-Fokker-Planck equation because it has an explicit non-local operator which acts solely on the velocities of particles, hence it does not conflict with the spatial confinement that we introduce. We conclude this introduction with a statement of the results we will present in the following chapters of this thesis and the resulting perspectives.

I.1 Classical diffusion equations

We start this introduction with a rather historical – if not always chronological – presentation of how our understanding of classical diffusion evolved throughout history. We begin with Fourier's derivation of the heat equation via the characterisation of the current density. Then, we explain how Brown's observation of pollen grains suspended in a fluid eventually led Einstein and Langevin (among others) to the mechanical explanation of the diffusion process through molecular motions at the microscopic level. This leads us to the introduction of the kinetic theory of gases and the mesoscopic description of a cloud of particles. In particular, we show how Langevin's approach gives rise to the Fokker-Planck equation and we conclude this first part of the introduction with the derivation of Fourier's law and the heat equation as a limit of kinetic equations.

I.1.1 The heat equation

I.1.1.1 Fourier's law and derivation of the heat equation

The oldest and most fundamental diffusion equation is the heat equation which describes the diffusion of heat through a medium. It was first derived by Fourier in 1822 in his seminal work *"Théorie analytique de la chaleur"* [Fou22]. In the first part of this book, he explains, based on several elaborate experiments, that:

> *"Pour connaître le flux actuel de la chaleur en un point p d'une droite tracée dans un solide, dont les températures varient par l'action des molécules, il faut diviser la différence des températures de deux points infiniment voisins du point p par la distance de ces points. Le flux est proportionnel au quotient."*
> In order to know the actual flux of heat at a point p on a line drawn through a solid, whose temperatures vary under the action of molecules, one must divide the difference in temperature of two infinitely close neighbours of the point p by the distance between those points. The flux is proportional to the quotient.

This law, referred to as **Fourier's law**, is the first step towards the derivation of the heat equation. This relation has been derived in several other fields of physics where diffusion phenomena may be observed, highlighting the ubiquity of Fourier's approach.

For instance, in 1827, Ohm established in his pioneer work [Ohm27] on electrodynamics a similar relation, called Ohm's law, which reads: the current through a conductor between two points is directly proportional to the voltage across those two points. Furthermore, another remarkable analogous relation – which will be particularly relevant in the context of this thesis – is Fick's law, derived by Fick in 1855 in the context of fluid dynamics in his work "On liquid diffusion" [Fic55], which states that:

> *"the transfer of salt and water occurring in a unit of time, between two elements of space filled with differently concentrated solutions of the same salt, must be, coeteris paribus, directly proportional to the difference of concentration, and inversely proportional to the distance of the elements from one another."*

To see how we can derive the heat equation from these laws, let us consider the context of fluid mechanics. We introduce a function $\rho(t, x)$, called the **particle density**, which describes the distribution of particles in a fluid in $\mathbb{R}^d$ in the sense that in an infinitesimal volume $\mathrm{d}x$ centred at x, there are $\rho(t, x)\,\mathrm{d}x$ particles at time t. Then, for any times $t_1 < t_2$ and any ball B in $\mathbb{R}^d$ the quantity:

$$\int_B \rho(t_2, x)\,\mathrm{d}x - \int_B \rho(t_1, x)\,\mathrm{d}x$$

is equal to the amount of particles that entered the ball B between times t_1 and t_2 minus the amount of particles that exited the ball during that same interval of time. Fourier's approach consists in equating this quantity with the number of particles that went through the surface ∂B of the ball between t_1 and t_2. This leads naturally to the notion of **current density vector** which represents the flux of particles through a surface, similarly to the heat flux of Fourier. We define the current density $J = J(t, x) \in \mathbb{R}^d$ by the following implicit relation: for any element of surface $\mathrm{d}S(x)$ centred at $x \in \partial B$ and oriented by the unit normal vector $n(x)$:

$$N_+ - N_- = J(t, x) \cdot n(x)\,\mathrm{d}S(x)\,\mathrm{d}t$$

where $N_\pm$ is the number of particles that crossed the element of surface $\mathrm{d}S(x)$ in the direction $\pm n(x)$, between the time interval $[t, t + \mathrm{d}t]$. Thus, we have

$$\int_B \big(\rho(t_1, x) - \rho(t_2, x)\big)\,\mathrm{d}x = -\int_{t_1}^{t_2} \int_{\partial B} J(t, x) \cdot n(x)\,\mathrm{d}S(x)\,\mathrm{d}t. \tag{I.1}$$

Writing

$$\rho(t_1, x) - \rho(t_2, x) = \int_{t_1}^{t_2} \partial_t \rho(t, x) \, \mathrm{d}t$$

and

$$\int_{t_1}^{t_2} \int_{\partial B} J(t, x) \cdot n(x) \, \mathrm{d}S(x) \, \mathrm{d}t = \int_{t_1}^{t_2} \int_{B} \nabla_x \cdot J(t, x) \, \mathrm{d}x \, \mathrm{d}t$$

we see that

$$\int_{t_1}^{t_2} \int_{B} \big(\partial_t \rho(t, x) + \nabla_x \cdot J(t, x)\big) \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Since this is true for all sets of the form $[t_1, t_2] \times B$ where B is a ball in $\mathbb{R}^d$, this yields the **continuity equation**

$$\partial_t \rho(t, x) + \nabla_x \cdot J(t, x) = 0. \tag{I.2}$$

This fundamental equation expresses the local conservation of mass, a basic property that is naturally required for a diffusion model. However, this equation is not closed since J is also unknown, and this is where Fourier's law – or rather **Fick's law** in this context – comes into play. Indeed, as explained above, this law states that for any time t and position x in the fluid, there exists $D > 0$ such that

$$J(t, x) = -D \nabla_x \rho(t, x). \tag{I.3}$$

Putting together (I.2) and (I.3) we get the **diffusion equation**

$$\partial_t \rho(t, x) - \nabla_x \cdot \big(D \nabla \rho(t, x)\big) = 0. \tag{I.4}$$

In the context of heat diffusion, D represents the thermal conductivity of the material and it is therefore natural, if the material is homogeneous, to assume that D is independent of x in which case we obtain the **heat equation**

$$\partial_t \rho(t, x) - D \Delta_x \rho(t, x) = 0. \tag{I.5}$$

I.1.1.2 Analysis of the heat equation in $\mathbb{R}^d$

Fourier constructed solutions of the heat equation in [Fou22] through a method of separation of variables and the development of Fourier series which allowed him to generate solutions in terms of infinite trigonometric series. He also constructed solutions in the form of integrals, developing the celebrated Fourier transform. Although these methods are "perhaps the most powerful and most daunting aspects of Fourier's work" according to Narasimhan in his excellent review of the tremendous influence of Fourier's work [Nar99], we give here some elements of a more modern analysis of the heat equation set on the whole space $\mathbb{R}^d$ which reads

$$\begin{cases} \partial_t \rho - D\Delta\rho = 0 & (t,x) \in [0,T) \times \mathbb{R}^d \\ \rho(0,x) = \rho_{in}(x) & x \in \mathbb{R}^d \end{cases} \tag{I.6}$$

for some initial condition ρ_{in}.

I.1.1.2.1 Scaling invariance and fundamental solution

One of the most crucial characteristics of the heat equation, and the diffusion process it models, is a particular scaling invariance which will be of importance throughout the rest of this chapter. Consider a solution ρ of the heat equation and define, for any $a \in \mathbb{R}$, the rescaled function ρ_a:

$$\rho_a(t,x) = a^d \rho(a^2 t, ax). \tag{I.7}$$

Then ρ_a also satisfies (I.5), indeed :

$$\partial_t \rho_a - D\Delta\rho_a = a^d \big(a^2 \partial_t \rho(a^2 t, ax) - D a^2 \Delta\rho(a^2 t, ax) \big) = 0.$$

Moreover, ρ_a has the same mass as ρ. Indeed we mentioned before that the continuity equation, and henceforth the heat equation, expresses the conservation of mass for the particle density and this constant mass is the same for ρ_a and ρ as can be seen easily by an integration of the equation and simple change of variable.

The scaling invariance motivates the following construction of solutions to the heat equation. In the one-dimensional case, if we choose $a = 1/\sqrt{Dt}$ and defined a function

ρ_a like we did in (I.7) then we see that we can define $G : \mathbb{R} \mapsto \mathbb{R}$ such that

$$\rho_a(t,x) = \frac{1}{\sqrt{Dt}}\rho\Big(\frac{1}{D}, \frac{x}{\sqrt{Dt}}\Big) = \frac{1}{\sqrt{Dt}}G\Big(\frac{x}{\sqrt{Dt}}\Big) = \frac{1}{\sqrt{Dt}}G(y)$$

where we introduced the **similarity variable**

$$y = \frac{x}{\sqrt{Dt}}.$$

The heat equation on ρ_a then reduces to a second degree ODE on G:

$$2G'' + yG' + G = \big(2G' + yG\big)' = 0.$$

As a consequence, $2G' + yG$ must constant and we will assume that this constant is 0 in order to find explicitly

$$G(y) = Ce^{-y^2/4}$$

for some constant C that we determine through the conservation of mass. Assuming ρ is of mass 1:

$$1 = \int_{\mathbb{R}} \rho(t,x)\,\mathrm{d}x = \frac{C}{\sqrt{Dt}}\int_{\mathbb{R}} e^{-\frac{x^2}{4Dt}}\,\mathrm{d}x = C\int_{\mathbb{R}} e^{-u^2}\,\mathrm{d}u = 2C\sqrt{\pi}.$$

We have constructed the following normalised fundamental solution of the heat equation

$$\Phi(t,x) := \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}}.$$

and extending this construction to higher dimension leads to an important object in the analysis of the heat equation:

Definition I.1.1. *The **fundamental solution** of the heat equation (also called **heat kernel**) Φ is defined on $(0,+\infty)\times\mathbb{R}^d$ as*

$$\Phi(t,x) = \frac{1}{(4\pi Dt)^{\frac{d}{2}}}e^{-\frac{|x|^2}{4Dt}}. \tag{I.8}$$

It is a Gaussian distribution centred around 0 with standard deviation $\sqrt{Dt}$ and it is a solution of

$$\begin{cases} \partial_t \Phi - D\Delta\Phi = 0 & (t,x) \in [0,T) \times \mathbb{R}^d \\ \Phi(0,x) = \delta_{x=0} & x \in \mathbb{R}^d \end{cases}$$

where δ is the Dirac delta function. It allows us to construct explicit solutions to the global Cauchy problem for the heat equation as follows:

Theorem I.1.1. *If we consider $\rho_{in} \in \mathcal{S}'(\mathbb{R}^d)$ then the associated heat equation* (I.6) *has a unique solution $\rho \in \mathcal{C}^\infty([0,+\infty); \mathcal{S}'(\mathbb{R}^d))$ given by the convolution of the initial datum ρ_{in} and the heat kernel Φ:*

$$\rho(t,x) = \rho_{in} * \Phi(t,x) = \int_{\mathbb{R}^d} \rho_{in}(y)\Phi(t,x-y)\,\mathrm{d}y. \tag{I.9}$$

We refer to [Tay11] for more details and the complete proof of this theorem. Note however that the uniqueness is only asserted within the class $\mathcal{C}^\infty([0,+\infty); \mathcal{S}'(\mathbb{R}^d))$ which entails bounds on the solution near infinity. If we look, instead, for solutions in $\mathcal{C}^1([0,+\infty); \mathcal{C}^\infty(\mathbb{R}^d))$ without any bounds on the growth at infinity, then we loose the uniqueness as was thoroughly investigated in 1-dimension by Tychonoff [Tyc35]. However, we can recover uniqueness of solution with an explicit extra constraint. For instance, Tychonoff showed that there is a unique solution ρ in $\mathcal{C}^1([0,+\infty); \mathcal{C}^\infty(\mathbb{R}^d))$ whose derivatives at all orders are bounded by some $M > 0$:

$$\left| \frac{\partial^n \rho}{\partial x^n} \right| < M.$$

Moreover, there is another extra condition, extremely relevant from a physical point of view, that has received the attention P.Rosenbloom and D.Widder in [Wid44] and [RW59] which is the non-negativity:

$$u(t,x) \geq 0. \tag{I.10}$$

They were able to show in 1-dimension that this condition ensures uniqueness in $\mathcal{C}^1([0,+\infty); \mathcal{C}^2(\mathbb{R}^d))$, and more precisely that the unique solution of the heat equation (I.6) that is always non-negative is precisely the one given by the convolution with the fundamental solution (I.9). This result was generalised by D.Aronson [Aro68] [Aro71] to higher dimensions and for a large class of parabolic PDEs.

The literature on the heat equation is quite extensive and there are many more interesting results on the subject, we refer e.g. to [Eva10] or [Tay11] and references within for more information.

I.1.2 Microscopic description of diffusion

I.1.2.1 The Brownian motion

A few years after Fourier's derivation of the heat equation, while Ohm was applying his approach to electrodynamics, another significant discovery was made by Scottish botanist R. Brown [Bro28] when observing the behaviour of pollen grains suspended in a liquid. He noticed that the grains were in a continuous motion that could not be accounted for by currents in the fluid, which led to the first description of what is now called the **Brownian motion**, illustrated in Figure I.1. Although the scientific community, at first, favoured the possibility that this motion was an evidence of life itself, Brown went on to observe a similar behaviour in inorganic, hence non-living, particles such as sand in a fluid, invalidating this possibility. The explanation for this behaviour came with the development of Kinetic theory in the second half of the XIXth century, which we will present in more detail in Section I.1.3. It describes the Brownian motion as the result of an enormous amount of microscopic particles that constitute the fluid colliding with the pollen grains which, although much bigger than the fluid particles, are still small enough for their motion to be affected by these collisions.

When this explanation was put forward, it was far from being unequivocally accepted by the whole scientific community, not only because the atomic theory – according to which matter, for instance fluid, is made of a plethora of extremely small atoms and molecules – was still the subject of controversy in the community but also because the mathematical framework required to rigorously describe the motion observed by Brown – for instance the notions of random walks and Markov processes – had not yet been articulated. Indeed, those notions will only be properly defined in the early XXth century, in particular by Pearson, Lord Rayleigh and Markov as well as Weiner and Lévy who defined in the 1920s the Wiener processes and the Lévy processes of which the Brownian motion is an example.

It is precisely at the beginning of the XXth century, while the formalisation of random walks was in its beginning, that A. Einstein studied the Brownian motion from a physical point of view and was able, in 1905 [Ein05], to establish a link between Brownian motion and diffusion processes. More precisely, he exhibited a relation between the mean-square-displacement of Brownian particles and the diffusion coefficient of the heat equation that governs the particle density function $\rho(t, x)$.

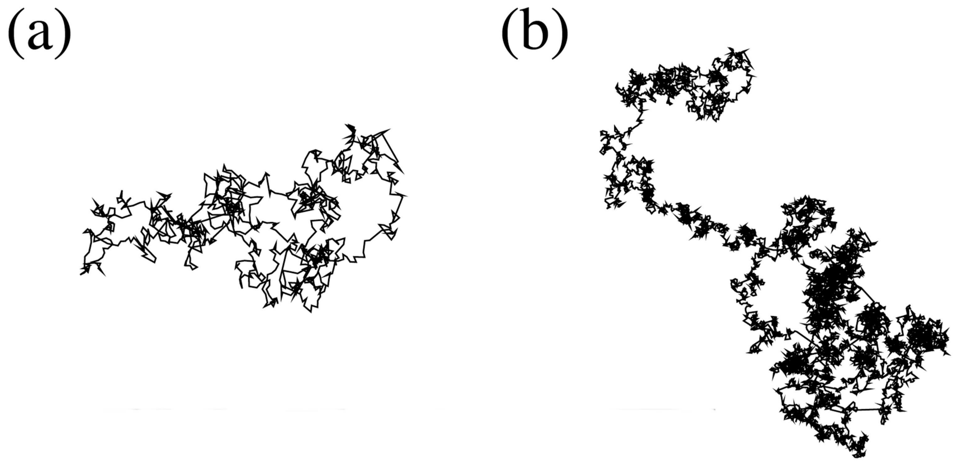


Fig. I.1 *Illustration of Brownian motion with 1000 steps (a) and 10000 steps (b)*

I.1.2.1.1 Diffusion approximation for Brownian motion

We consider $\rho(t,x)$ the particle density defined in the previous section, in the one-dimensional case: $x \in \mathbb{R}$, and introduce the probability density function $\lambda(\Delta x)$ for a jump of length $|\Delta x|$ with $\Delta x \in \mathbb{R}$ (which of course had a rather imprecise definition in Einstein's paper since the notion of probability density was not yet invented). The idea is to compute the particle density at time $t+\Delta t$ for some small $\Delta t > 0$ using that fact that a particle can be at position x at time $t+\Delta t$ only if it was at position $x-\Delta x$ at time t and made a jump of length Δx. Einstein formulate this idea through the following integral equation:

$$\rho(t+\Delta t, x) = \int_{\mathbb{R}} \rho(t, x-\Delta x)\lambda(\Delta x)\,\mathrm{d}\Delta x. \tag{I.11}$$

If Δt is small enough then, in first approximation, we have

$$\rho(t+\Delta t, x) = \rho(t,x) + \Delta t \partial_t \rho(t,x) + o(\Delta t)$$

and furthermore, if we assume that λ "differs from zero for very small values of Δx only" [Ein05] then it makes sense to write an expansion in orders of Δx:

$$\rho(t, x+\Delta x) = \rho(t,x) + \Delta x \partial_x \rho(t,x) + \frac{\Delta x^2}{2}\partial^2_{xx}\rho(t,x) + o(\Delta x^2).$$

Since only small values of Δx contribute to the integral above, we can perform this expansion under the integral in (I.11). Moreover, since we assumed that λ only depends on the length $|\Delta x|$ we have

$$\langle \Delta x \rangle := \int_{\mathbb{R}} \Delta x \lambda(\Delta x)\,\mathrm{d}\Delta x = 0$$

since the integrand is an odd function. Hence, if we only consider the highest order term, (I.11) yields

$$\Delta t \partial_t \rho(t,x) + o(\Delta t) = \int_{\mathbb{R}} \left(\frac{\Delta x^2}{2} \partial^2_{xx} \rho(t,x) + o(\Delta x^2) \right) \lambda(\Delta x)\,\mathrm{d}\Delta x$$

so we recover in first approximation the heat equation

$$\partial_t \rho(t,x) = D \partial^2_{xx} \rho(t,x)$$

where

$$D = \frac{1}{2\Delta t} \langle \Delta x^2 \rangle := \frac{1}{2\Delta t} \int_{\mathbb{R}} \Delta x^2 \lambda(\Delta x)\,\mathrm{d}\Delta x.$$

Earlier in the same article [Ein05], Einstein computed this diffusion coefficient D by studying the particle density ρ when the system reaches thermodynamic equilibrium. He wrote explicitly the conditions ρ must satisfy in order for the diffusion force and the friction force in the fluid to perfectly balance each other (dynamic equilibrium) and, furthermore, for the energy to be constant throughout the system (thermal equilibrium). Introducing the temperature T, the pressure P, the ideal gas constant R, the number of particles N, the radius of the spherical particles considered r and the viscosity of the fluid μ, he proved that the diffusion constant must be

$$D = \frac{RT}{N} \frac{1}{6\pi r \mu}.$$

Together, the two formulae yield the following relation between the **mean-square-displacement** $\langle \Delta x^2 \rangle$ and the time step Δt:

$$\langle \Delta x^2 \rangle = \frac{RT}{N} \frac{1}{3\pi r \mu} \Delta t. \tag{I.12}$$

This formula was of particular interest to the scientific community because there was good hope to be able to measure this mean-square-displacement and so from this relation it becomes possible to estimate not only the true size of atoms but also the Avogadro number, i.e. the number of atoms in a mole. This was indeed eventually obtained experimentally by Perrin in 1910 which allowed him to compute the Avogadro number and thus consolidate the atomic theory.

Remark I.1.2. *Notice how* (I.12) *illustrates the scaling property of the heat equation that we presented in* (I.7). *If one rescales the space and time steps by some $a > 0$ as*

$$\Delta x \to a\Delta x \quad \text{and} \quad \Delta t \to a^2\Delta t$$

then the relation remains unchanged.

I.1.2.1.2 The Wiener process

The formal definition of a Wiener process, the stochastic description of the Brownian motion, reads as follows

Definition I.1.2. *A Wiener process W_t is a stochastic process characterised by the following conditions*

i) $W_0 = 0$ *almost surely*

ii) W_t *has continuous paths, i.e. it is almost surely continuous with respect to t*

iii) W_t *has independent increments: for all $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$, the random variables $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent.*

iv) W_t *has stationary Gaussian increments: for all $t, s \geq 0$, $W_{t+s} - W_t$ is normally distributed with mean 0 and variance s*

$$W_{t+s} - W_t \sim \mathcal{N}(0, s)$$

where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

The first incomplete definition of such processes was given in 1900 by Bachelier [Bac00], a student of H. Poincaré, in the field of stock market speculations, without any explicit mention of the Brownian motion. The rigorous proof of existence of such processes as a limit of discrete random walks was established through several different methods, first by Wiener in 1923 [Wie23] by introducing the Wiener measure on the space of continuous functions on $[0, 1]$, then by Wiener again [Wie24] using Fourier series, then in 1931 by Kolmogoroff [Kol93] who gave a rigorous version of Bachelier's argument based on Gaussian martingales, and also by Lévy in 1940 [Lév80] by an interpolation argument. The proof of Lévy is very common in modern literature on the subject, the idea is to build a process which satisfies *i)*, *iii)* and *iv)* on the set $\mathcal{D}_n$ of dyadic numbers in $[0, 1]$:

$$\mathcal{D}_n = \left\{ \frac{k}{2^n} : 0 \leq k \leq 2^n \right\}$$

then take the limit as n goes to infinity to get a continuous process on $[0, 1]$, extend it to $[0, +\infty)$ and, finally, check that this extended limit, which is time-continuous by construction, still satisfies *i)*, *iii)* and *iv*).

I.1.2.2 The Langevin equation

The linear characterisation of the mean-square-displacement (I.12) was also derived, three years after Einstein, by Langevin with a demonstration that was "infinitely more simple by means of a method that is entirely different" [Lan08] to quote his own words. Indeed, Langevin's method differs drastically from Einstein's because it is purely microscopic. Instead of using the Brownian motion to describe the evolution of the particle density ρ, he uses it to describe the average motion of a particle (polen grain) in a fluid as a result of external forces, building on the work of Smoluchowski [Smo16]. The starting point is Newton's second law of motion applied to the position $x(t)$ of a particle in the fluid. Using Stokes' formula according to which the viscosity force on the particle is $-6\pi\mu r \frac{\mathrm{d}x}{\mathrm{d}t}$, this yields

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -6\pi\mu r \frac{\mathrm{d}x}{\mathrm{d}t} + X \tag{I.13}$$

where μ is the viscosity of the fluid, r is the radius of the spherical particle and X is a "complementary force that is indifferently positive and negative and its magnitude is such that it maintains the agitation of the particle, which the viscous resistance would stop without it".

Equation (I.13) is the first example of a wide class of stochastic differential equations called the **Langevin equation**, although it is usually written for the velocity variable $v(t) = \mathrm{d}x/\mathrm{d}t$. In his paper, Langevin did not need to identify the process X precisely, he only cared that its expected value $\mathbb{E}(X)$, or as he calls it "the average value of the term $Xx(t)$", be zero which he justifies by the "irregularity of the complementary forces X" and is indeed satisfied for a Wiener process, c.f. *iv)* in Definition I.1.2. Langevin multiplies the equation by $x(t)$ to get

$$\frac{m}{2}\frac{\mathrm{d}^2x^2}{\mathrm{d}t^2} - m\Big(\frac{\mathrm{d}x}{\mathrm{d}t}\Big)^2 = -3\pi\mu r\frac{\mathrm{d}x^2}{\mathrm{d}t} + Xx \tag{I.14}$$

and looks at what this equation entails when averaged over the vast number of particles in the fluid. If the average motion $\bar{x}$ satisfies (I.14) then Langevin recognizes the kinetic energy in the second term on the left-hand-side which is given by the thermal equilibrium relation as:

$$m\Big(\frac{\mathrm{d}\bar{x}}{\mathrm{d}t}\Big)^2 = \frac{RT}{2N}$$

where N is the number of particles, R is the ideal gas constant and T is the temperature. Introducing the mean-square-displacement $z(t) = \mathrm{d}\bar{x}^2/\,\mathrm{d}t$, he obtains:

$$\frac{m}{2}\frac{\mathrm{d}z}{\mathrm{d}t} + 3\pi\mu r z(t) = \frac{RT}{N}.$$

He then solves this ODE for $z(t)$ which reads, for some constant C, as

$$z(t) = \frac{RT}{N}\frac{1}{3\pi\mu r} + Ce^{-\frac{6\pi\mu r}{m}t}$$

and he recovers Einstein's formula (I.12) in the "long-time" asymptotic. Note that the coefficient in the exponential is of order 10^{-8} so the "long-time" is actually t greater than 10^{-8} seconds.

Langevin's work had tremendous influence over the subsequent development of kinetic theory and stochastic calculus. Indeed, he makes the first step towards the notion of **Gaussian white noise** with his force X, which he presents as a stochastic perturbation of Newton's second law of motion, hence introducing the stochastic equivalent of Newton's second law: the Langevin equation, making him the founder of the field of Stochastic Differential Equations. The Langevin equation, and stochastic PDEs in general, are now widely used to model non-deterministic phenomena, we refer to [CKW96] for more information on the implications of Langevin's work.

I.1.3 Kinetic equations

I.1.3.1 Introduction to kinetic theory

Kinetic theory was first developed in the second half of the XIXth century. Although the pioneer works of Bernouilli, Clausius, Krönig and several others had a strong impact on its original development, the invention of kinetic theory is accredited to L. Boltzmann and J.C. Maxwell

Kinetic theory embodies a link between the microscopic description of a fluid (gas, plasma, or any large cloud of particles) where we describe the movement of a single particle using Newton's laws of motion – or, as we have seen above, the Brownian motion or Langevin equations – and the macroscopic description of a fluid with models such as the heat equation, the Euler equation or the Navier-Stokes equation for example. Indeed, as we have seen in the previous sections, when we derive the heat equation from the Brownian motion, we used a Taylor expansion and only kept the highest order terms, loosing all the information hidden in the lower order ones. One of the purpose of kinetic equations is then to provide us with a scale of observation which retains all the information of the microscopic description and, at the same time, allows us to investigate macroscopic characteristics of the fluid. The first, and most fundamental, illustration of this idea is the Maxwell-Boltzmann distribution:

$$M(v) = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}} \tag{I.15}$$

where m is a particle's mass, T is the thermodynamic temperature and k is Boltzmann's constant. It describes the distribution of the speeds of particles in idealised gases at equilibrium and it connects the microscopic and macroscopic scales since its variable v is inherently microscopic and it expresses how the velocities are distributed in the whole macroscopic gas. It was heuristically derived by Maxwell in 1867 [Max67] and Boltzmann proved in 1872 [Bol95] through his celebrated H-theorem that gases should over time tend toward this distribution of velocities.

The key concept that led Maxwell and Boltzmann to the elaboration of kinetic theory is the idea that all measurable quantities, i.e. all observable macroscopic characteristics of a fluid, can be expressed in terms of microscopic averages. To illustrate this idea, let us introduce the unknown of the kinetic equations we will consider: the **probability density function** $f(t,x,v)$. It depends on time $t \in [0,T)$ or $[0,+\infty)$, position $x \in \Omega \subseteq \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$. For any fixed time t, the quantity $f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v$ represents the density of particles in an infinitesimal volume $\mathrm{d}x\,\mathrm{d}v$ of the phase-space or, in other words, the density of particles in a volume $\mathrm{d}x$ centred at $x \in \Omega$ and a velocity v' in the neighbourhood $\mathrm{d}v$ of v. By the averaging of the distribution function f, we can recover macroscopic quantities such as the particle density $\rho(t,x)$, the macroscopic velocity $u(t,x)$ or the local temperature $T(t,x)$ as follows:

$$\begin{aligned} &\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad \rho(t,x)u(t,x) = \int_{\mathbb{R}^d} v f(t,x,v)\,\mathrm{d}v, \\ &\rho(t,x)|u(t,x)|^2 + d\rho(t,x)T(t,x) = \int_{\mathbb{R}^d} |v|^2 f(t,x,v)\,\mathrm{d}v. \end{aligned} \tag{I.16}$$

The evolution of the distribution function f is deduced from the microscopic description of the motion of particles as we will present now.

I.1.3.1.1 Collisionless setting

Let us start with the simplest situation, which is the collisionless setting. We look at a cloud of particles where the particles do not interact with each other. In first approximation, let us assume that there is no friction, or viscosity, phenomenon. In this setting, a particle will move in a straight line with constant velocity. The position and velocity of a particle $(x(t), v(t))$ will then satisfy $\frac{\mathrm{d}x}{\mathrm{d}t} = v$, $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$. If we differentiate the distribution f along those characteristic lines we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(t, x(t), v(t)\big) = 0$$

which yields the **free transport equation**, also called **Vlasov equation**:

$$\partial_t f + v \cdot \nabla_x f = 0. \tag{I.17}$$

Given an initial condition

$$f(0,x,v) = f_{in}(x,v) \tag{I.18}$$

an explicit solution is

$$f(t,x,v) = f_{in}(x - tv, v).$$

If there is a macroscopic force E acting on the particles then their trajectories will not be straight lines anymore. Newton's second law of motion states that $m\frac{\mathrm{d}v}{\mathrm{d}t} = E$ which yields the kinetic equation (rename $E := E/m$):

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0.$$

In particular, E could be an external electric field: $E = E(t,x)$ independent of f, or E could model the self-consistent electric field generated by the particles in which case we get the celebrated **Vlasov-Poisson equation**

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0, \\ E(t,x) = \nabla_x \phi(t,x), \qquad \Delta_x \phi = \rho(t,x). \end{cases}$$

These are just two examples of macroscopic forces, one can also consider a more complex self-consistent electro-magnetic field given by a coupling with Maxwell's equations of electromagnetic. This model is particularly relevant, for instance, when considering quasineutral plasmas and laser-plasma interactions. It has received a lot of attention, we refer e.g. to [GS86], [BMP03], [BGP03] or [CL06] for more information.

I.1.3.1.2 Collisional setting, the Boltzmann operator

Now, let us assume again that there are no macroscopic forces but let us consider the collisional setting. In order to derive the kinetic equation that governs the evolution of f we need to model the collisions between particles. There are several ways to model these collisions, we present here one of the most celebrated models: the Boltzmann operator, and we devote Section I.1.3.2 to another remarkable model: the Fokker-Planck operator.

The Boltzmann collision operator was first derived heuristically by Maxwell in [Max67] and then formalised by Boltzmann in 1872 [Bol95] using the following structural assumptions:

1. Binary collisions: this comes down to assuming that the gas is dilute enough to assume that the occurrence of a collision of 3 or more particles simultaneously is rare enough to be neglected.

2. Localised collisions: we assume collision are brief events, localised both in time and space, meaning that the duration of collision is assumed to be very small with respect to the time scale that we consider.

3. Elastic collisions: momentum and kinetic energy are preserved in the collision process. This assumption yields the following relations for the collision of two

particles of velocity v' and v'_* which acquire the velocities v and v_* respectively after collision:

$$\begin{cases} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \end{cases}$$

4. Microreversible collisions: we assume the microscopic dynamics are time-reversible which means that the probability of a pair of velocity (v', v'_*) to become (v, v_*) after collision is the same as the probability of (v, v_*) to become (v', v'_*).

5. Boltzmann's chaos assumption: we assume that the velocities of two particles before collision are uncorrelated. This assumption is of great importance in the field of kinetic theory and is the subject of many works, it implies an asymmetry between past and future which plays a crucial role in one of the most fundamental questions of kinetic theory: the loss of reversibility when we consider a large number of reversible dynamics.

These assumptions were stated by Boltzmann in 1872 [Bol95] and allowed him to formally derive the Boltzmann operator to model the collisions of particles at a kinetic scale. Note that the rigorous derivation of the Boltzmann operator from Newton's law of motion is still an open problem although it has been proven for very small time, smaller than the mean time of the first collision. From the elastic collision assumption, we deduce the σ-representation where $\sigma \in \mathbb{S}^{d-1}$ denotes

$$\sigma = \frac{v' - v'_*}{|v' - v'_*|}$$

with which the elastic relation can be written as

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma. \end{cases}$$

Introducing the notation $f' = f(t, x, v')$, $f_* = f(t, x, v_*)$ and $f'_* = f(t, x, v'_*)$, the Boltzmann operator then take the form

$$Q(f, f) = \iint_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \cos\theta)\big(f' f'_* - f f_*\big)\,\mathrm{d}v_*\,\mathrm{d}\theta \tag{I.19}$$

where B is the Boltzmann collision kernel and encodes the physics behind the collisions process and $\cos\theta$ is the scalar product $\frac{v - v_*}{|v - v_*|} \cdot \sigma$. The kernel B can be expressed in several different ways and we refer e.g. to [Cer88], [FS02] and references within for detailed examples. What remains invariant, whatever the kernel we consider, is that the **Boltzmann operator** can formally be expressed as a difference of two terms: Gain and Loss

$$Q(f,f)(t,x,v) = Q_+(f,f)(t,x,v) - Q_-(f,f)(t,x,v)$$

where the gain term Q_+ represents the number of particles that had a velocity v' and acquired the velocity v as a result of collisions and the loss term Q_- represents the number of particles that had velocity v and lost it, to acquire another velocity v' after collisions. Note however that in many cases both Q_+ and Q_- are infinite when written separately, which is why this formulation is formal. Nevertheless, it motivates a simpler version of this collision operator: the linear Boltzmann operator, which preserves this structure of gain and loss but through a linear interaction instead of the bilinear operator written above. Considering a non-negative collision kernel $\sigma = \sigma(v,v')$, the **linear Boltzmann operator** L_B reads

$$L_B(f) = \int_{\mathbb{R}^d} \Big(\sigma(v,v')f' - \sigma(v',v)f \Big) \, \mathrm{d}v' \tag{I.20}$$

where $\sigma(v,v')$ represents the probability for a particle with velocity v to acquire the velocity v' after collisions. Note that this operator is not to be confused with the linearised Boltzmann operator, which is the linearisation of $Q(f,f)$ around its equilibrium.

In this thesis, our focus does not lie in the study of the linear Boltzmann operator but we will present a few results on a particularly simple version of this operator because of their significant influence. This simple case is sometimes called the **linear relaxation operator** and corresponds to the linear Boltzmann with $\sigma(v,v') = M(v)$ where M is the **local Maxwellian**

$$M(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}} \tag{I.21}$$

which yields

$$L(v) = \rho(t,x)M(v) - f(t,x,v). \tag{I.22}$$

Although quite simplistic, this operator conserves the relaxation property of the Boltzmann equation which is crucial in our analysis, as we will see in section I.1.3.4. The kinetic equation associated with this operator takes the form of a balance between the free-transport and collision. It reads

$$\partial_t f + v \cdot \nabla_x f = \rho(t,x)M(v) - f(t,x,v) \tag{I.23}$$

and is often called the **Vlasov-linear relaxation equation**.

I.1.3.2 The Fokker-Planck and Vlasov-Fokker-Planck equations

The Fokker-Planck equation was first derived by Fokker in 1914 [Fok14] and Planck in 1917 [Pla17] to describe the evolution of the velocities of particles in a fluid. It was independently discovered by Kolmogoroff in 1931 [Kol93] through a significantly different method, which is why it is sometimes referred to as the **Kolmogoroff forward equation**, and it was applied by Smoluchowski [Smo16] to particle diffusion in which case it is called the **Smoluchowski equation**.
The Fokker-Planck equation expresses a balance between a drift and a diffusion force, much like the Langevin equation dissociates Stokes' viscosity from the Brownian motion. We will show how it can be derived by a generalisation of Einstein's diffusion approximation. Note that we do not follow the original derivation of Fokker and Planck but, instead, one based on the works of Kramers [Kra40], Moyal [Moy49] and Pawula [Paw67].

I.1.3.2.1 Derivation of the Fokker-Planck equation

We consider a velocity probability density $W(t,v)$ – equivalent of the particle density $\rho(t,x)$ but for the velocities – which describes the velocity distribution in a fluid. In order to generalise Einstein's approach, we define a conditional transition probability $\lambda(t+\Delta t, v|t, v')$ that a particle with velocity v' at time t acquires a velocity v at time $t+\Delta t$. The integral relation (I.11) then reads

$$W(t+\Delta t, v) = \int_{\mathbb{R}^d} W(t, v-\Delta v)\lambda(t+\Delta t, v|t, v-\Delta v)\,\mathrm{d}\Delta v. \tag{I.24}$$

Remark I.1.3. *Note that the most general form of this equation would be to consider that λ depends on the velocities at all the previous times $t-k\Delta t$, $0 \le k \le t/\Delta t$. We implicitly assumed here that λ has no memory in the sense that it only depends on the velocity at times t. This is equivalent to assuming that a process with transition probability λ satisfies the Markovian property. We refer to [Ris96, Section I.2.4.1] for more details.*

We introduce the notation $M_n(v - \Delta v, t, \Delta t)$, $n \geq 0$ for the moments of the transition probability as a function Δv defined as

$$M_n(t, \Delta t, v) = \int_{\mathbb{R}^d} (\Delta v)^n \lambda(t + \Delta t, v + \Delta v | t, v)\, \mathrm{d}\Delta v. \tag{I.25}$$

Assuming we know those moments, we can write the Taylor expansion of the integrand in (I.24) as

$$\begin{aligned} W(t, v - \Delta v)\lambda(t + \Delta t, v | t, v - \Delta v) &= W(t, v - \Delta v)\lambda(t + \Delta t, v - \Delta v + \Delta v | t, v - \Delta v) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\Delta v)^n \frac{\mathrm{d}^n}{\mathrm{d}v^n} \Big[W(t, v)\lambda(t + \Delta t, v + \Delta v | t, v) \Big] \end{aligned}$$

and integrating with respect to Δv (assuming the necessary convergence of the series and the moments) this yields using (I.24) on the left-hand-side, and (I.25) on the right-hand-side

$$W(t + \Delta t, v) = \sum_{n=0}^{\infty} \Big(-\frac{\mathrm{d}}{\mathrm{d}v} \Big)^n \left[\frac{M_n(t, \Delta t, v)}{n!} W(t, v) \right].$$

Now, we first notice that since λ is a probability density, $M_0(t, \Delta t, v) = \int \lambda \, \mathrm{d}\Delta v = 1$. For the moments M_n, $n \geq 1$ we want to do a first order Taylor expansion with respect to time, assuming Δt is small. Since we have obviously

$$\lambda(t, v | t, v') = \delta_{v=v'}$$

for all v, v', the order 0 term in the Taylor expansion will be null for all M_n. Hence, we define the expansion coefficients $m_n(t, v)$ by the implicit relation

$$\frac{M_n(t, \Delta t, v)}{n!} = m_n(t, v)\Delta t + O(\Delta t^2).$$

Putting the $n = 0$ term on the left-hand-side and dividing by Δt this yields

$$\frac{W(t + \Delta t, v) - W(t, v)}{\Delta t} = \sum_{n=1}^{\infty} \Big(-\frac{\mathrm{d}}{\mathrm{d}v} \Big)^n \Big[m_n(t, v) W(t, v) \Big] + O(\Delta t).$$

Finally, taking the limit as Δt goes to 0 we get the **Kramers-Moyal equation** for $W(t,v)$:

$$\partial_t W(t,v) = \sum_{n=1}^{\infty} \Big(-\frac{\mathrm{d}}{\mathrm{d}v} \Big)^n \Big[m_n(t,v) W(t,v) \Big].$$

Remark I.1.4. *It goes without saying that the derivation of the Kramers-Moyal equation that we just presented is quite formal and one would need to control the convergence of the sums and integrals in order to make it rigorous but that is not our purpose here. We refer to [Ris96, Chapter 4] for more details on this derivation, as well as the original papers of Kramers [Kra40] and Moyal [Moy49].*

In 1967, in an effort to justify the Fokker-Planck model, Pawula proved in [Paw67] by a subtle use of the generalised Cauchy-Schwarz inequality on the family of moments M_n, that given the assumption we made on λ, there are only three possibilities:

- All moments of order $n > 1$ are null
- All moments of order $n > 2$ are null
- An infinite number of moments are not null

Pawula was able to make an explicit link between those three situations and the underlying process described by the transition probability λ. Indeed, he proved that if the process is deterministic, i.e. no randomness: the particle moves right at every time step Δt with speed $m_1(t,v)$, then we are in the first situation with a hyperbolic transport equation:

$$\begin{cases} \partial_t W(t,v) = -\nabla_v \big[m_1(t,v) W(t,v) \big] \\ W(0,v) = W_{in}(v) \end{cases}$$

the solution of which if $m_1 = c$ constant is $W(t,v) = W_{in}(v - ct)$. Furthermore, if the underlying process is governed by a Langevin equation then we are in the second case and the moments m_1 and m_2 represent the viscosity and the diffusion coefficients $-\mu(t,v)$ and $D(t,v)$ respectively and we obtain the **general Fokker-Planck equation**

$$\partial_t W = \frac{\mathrm{d}}{\mathrm{d}v} \big[\mu v W \big] + \frac{\mathrm{d}^2}{\mathrm{d}v^2} \big[DW \big]. \tag{I.26}$$

In this thesis we will focus on the case where the viscosity and diffusion coefficients μ and D are constant in which case the **Fokker-Planck equation** reads

$$\partial_t W = \mu \nabla_v \cdot \big(vW\big) + D\Delta W. \tag{I.27}$$

Remark I.1.5. *The third case of Pawula's theorem may be useful in some cases, for instance to model Generation and Recombination processes, see e.g. [Ris96, Section I.4.5], but in those cases the transition probability must be allowed to take negative values, at least for small times, which is not very relevant when modelling fluids.*

I.1.3.2.2 Solutions of the Fokker-Planck equation in $\mathbb{R}^d$

Like the heat equation, the Fokker-Planck equation (I.27) is a linear parabolic PDE and admits a Green function i.e. a fundamental solution Φ_{FP} defined on $(0,+\infty)\times\mathbb{R}^d$ as

$$\Phi_{FP}(t,x) = \left(\frac{\mu}{2\pi D(1-e^{2\mu t})}\right)^{\frac{d}{2}} e^{-\frac{\mu|x|^2}{2D(1-e^{-2\mu t})}} \tag{I.28}$$

which is solution of the Fokker-Planck equation with localised initial datum

$$\begin{cases} \partial_t \Phi_{FP} = \mu\nabla_v \cdot (v\Phi_{FP}) + D\Delta_v \Phi_{FP} = 0 & (t,v) \in [0,T) \times \mathbb{R}^d \\ \Phi_{FP}(0,v) = \delta_{v=0} & v \in \mathbb{R}^d. \end{cases}$$

Analogously to the fundamental solution for the heat equation, Φ_{FP} allows us to construct global solutions for the Fokker-Planck equation with initial condition $W(0,v) = W_{in}(v) \in \mathcal{S}'(\mathbb{R}^d)$ by a convolution in v

$$W(t,v) = W_{in} * \Phi_{FP}(t,v).$$

I.1.3.2.3 Derivation of the Vlasov-Fokker-Planck equation

Fokker and Planck's approach can be extrapolated to define a collision operator at the kinetic scale, which will actually be the main focus of all the results we present in this thesis. The derivation of the kinetic Fokker-Planck equation differs from the derivation of the kinetic Boltzmann equation in the same way that Langevin's proof of the linearity of the mean-square-displacement differs from Einstein's. Instead of a gain-loss approach, we take a microscopic point of view and describe the position and velocity of a particle in the fluid by the random variables $(x(t), v(t))$ whose evolution is governed by a free-transport/Langevin equation:

$$\begin{cases} \dot{x} = v(t) \\ \dot{v} = -\mu v(t) + DB_t \end{cases} \tag{I.29}$$

where B_t is a Wiener process and the dot denotes the derivative in time. The first equation describes the free-transport of particles as presented earlier in the collisionless setting, and the second describes the evolution of the velocity as a balance between a friction force and a Brownian motion. The associated kinetic equation is called the **Vlasov-Fokker-Planck equation**, it reads

$$\partial_t f + v \cdot \nabla_x f = \mu \nabla_v \cdot \big(vf\big) + D\Delta_v f \tag{I.30}$$

and we will systematically assume $\mu = D = 1$ without loss of generality for the mathematical analysis of the equation. This equation is also sometimes called the Kramers equation in reference to Kramers' work including the derivation of the Kramers-Moyal equation, or also the Smoluchowski equation in the 1-dimensional case.

I.1.3.3 Some properties of collision operators

In the rest of this section, we will focus on the linear relaxation and the Fokker-Planck operator, right-hand-side of (I.23) and (I.30) respectively. Before we derive the heat equation from the associated kinetic models, which is the subject of the following section, we would like to present some fundamental properties that these operators have in common. The first and most obvious property is the conservation of mass which follows from

$$\int_{\mathbb{R}^d} L(f)\,\mathrm{d}v = 0$$

where L is either the linear relaxation or the Fokker-Planck operator. Note that this property follows directly from the definitions of our operators. Another crucial property is the existence and uniqueness of an equilibrium:

Proposition I.1.6. *Let L be either the linear relaxation* (I.22) *or the Fokker-Planck* (I.27) *operator, then there exists a unique normalised equilibrium M:*

$$\exists!\, M(v) \geq 0 \text{ on } \mathbb{R}^d, \qquad \int_{\mathbb{R}^d} M(v)\,\mathrm{d}v = 1, \qquad L(M) = 0$$

and this equilibrium is a local Maxwellian distribution

$$M(v) = \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2}. \tag{I.31}$$

This should be interpreted in the light of Boltzmann's H-theorem as an illustration of the compatibility of our models of collision and the Maxwell-Boltzmann equilibrium distribution of velocities. The third property which we will be central to the diffusion limit and is common to both operator, is their dissipativity:

Proposition I.1.7. *For any $f(x,v)$ regular enough, the* ***dissipation*** *$\mathcal{D}$, defined as*

$$\mathcal{D}(f) := - \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f L(f) \frac{\mathrm{d}x\,\mathrm{d}v}{M(v)}$$

where M is the equilibrium (I.31)*, satisfies*

$$\mathcal{D}(f) \geq C \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(f - \langle f\rangle M\big)^2 \frac{\mathrm{d}x\,\mathrm{d}v}{M(v)} \geq 0 \tag{I.32}$$

for some constant C independent of f, with $\langle f\rangle = \int f\,\mathrm{d}v$.

This is a crucial property of collision operators that is very useful for the physical justification of a kinetic equation. Indeed, we see in (I.32) that the dissipation controls the distance between the probability density f and the velocity-equilibrium state of the cloud of particle where the velocities are distributed according to the local Maxwellian (I.21). As a consequence, proving that the dissipation decreases towards 0 entails the convergence of the system towards a velocity-equilibrium state, in accordance with the second law of thermodynamics.
Note that in the Fokker-Planck case, the dissipation takes the form of a homogeneous H^1 norm in a weighted space:

$$\mathcal{D}_{FP}(f) = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Big|\nabla_v\Big(\frac{f}{M}\Big)\Big|^2 M\,\mathrm{d}x\,\mathrm{d}v$$

and since the weight is a local Maxwellian, we can use the Poincaré inequality to show (I.32). In the linear-relaxation case, on the other hand, (I.32) is actually an equality and the proof does not involve the gradient.
The Cauchy problem for kinetic equations with either a linear Boltzmann operator, including the linear-relaxation case (I.23), or a Fokker-Planck operator (I.30) with

or without a Poisson potential have received a great deal of interest throughout the years. We refer e.g. to the excellent books [Cer88] and [FS02] and references within for the Boltzmann case, and to [Bou93] and [CS95] for the Vlasov-Poisson-Fokker-Planck (VPFP) equation, as well as [VO90] where the authors construct global solutions by generalising a fundamental solution argument for the VPFP system. In the context of this thesis, the existence result that is the most relevant (although far from optimal) is the following.

Theorem I.1.8. *Consider an initial condition f_{in} such that*

$$\begin{cases} f_{in} \geq 0 \\ f_{in} \in L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d) \text{ where } M(v) \text{ is the equilibrium (I.31)} \end{cases}$$

Then the Cauchy problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = L(f) & (t,x,v) \in [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \end{cases} \tag{I.33}$$

where L is either a linear relaxation operator (I.22) or a Fokker-Planck operator (I.27), admits a weak solution $f \in \mathcal{C}^0([0,+\infty), L^1(\mathbb{R}^d \times \mathbb{R}^d))$ which satisfies

$$\begin{cases} f \geq 0 \\ f(t,\cdot,\cdot) \in L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d) \end{cases}$$

Note in particular that this notion of weak solution is physically relevant in the sense that it entails finite mass, kinetic energy and entropy.

I.1.3.4 Diffusion limit of kinetic equations

Since we know that the particle density ρ can be obtained by averaging the kinetic solution f of (I.33), as expressed in (I.16), it is only natural to wonder if we can derive the equations that govern ρ from kinetic equations. This is the subject of this section. For the Vlasov-Fokker-Planck equation, the answer to this question began with the pioneer works [Wig61], [LK74] and [HM75], was rigorously established in one-dimension in [DMG87], extended to two and three dimensions for small time interval in [PS00], long time interval in [Gou05] and to higher dimension in [EGM10].

The first thing we notice is that the unit of time and space that we implicitly used when deriving the Langevin equation, which we used again to derive the kinetic equation, is much smaller than the time and space scales that are naturally used for the heat equation. Indeed, the unit of time in the kinetic equation is linked to the time scale of the collision process which is of the order of the average time between two consecutive collisions of a particle, whereas in the macroscopic heat equation, a vast number of collisions happen per unit of time. Same goes for the unit of distance which, at the kinetic scale, is comparable to the **mean-free-path**: the average distance a particle travels between two collisions, whereas at the the macroscopic scale there are about 10^{23} particles in a "small" element of volume so the macroscopic unit of distance is much greater than the mean-free-path. Hence, to derive macroscopic equations from the kinetic ones we need to rescale time and space, and to that end we introduce the **Knudsen number** ε, named after Danish physicist Knudsen from the late XIXth early XXth century, formally defined as the ratio

$$\varepsilon = \frac{\text{mean-free-path}}{\text{considered length scale}}. \tag{I.34}$$

The Knudsen number formalises a continuum between the kinetic scale at $\varepsilon = 1$ and the macroscopic scale in the limit as ε goes to 0. We rescale the space variable x as

$$x' = \varepsilon x \tag{I.35}$$

and since our purpose is to derive a diffusion equation for ρ we will choose the rescaling of time that agrees with the time linearity of the mean-square-displacement (I.12), hence:

$$t' = \varepsilon^2 t. \tag{I.36}$$

This is called a parabolic scaling. Investigating the asymptotic behaviour, as ε goes to 0, of the resulting rescale kinetic equation and its rescaled solution f_ε, is usually called taking the **diffusion limit of the kinetic equation**. Other rescaling limits can be considered, such as the hyperbolic or hydrodynamical limit which allows e.g. to derive the Euler or the Navier-Stokes equations from the Boltzmann equation, and we refer to [MSR03], [JLM09] and references therein for more information on that topic.
With the parabolic scaling, the rescaled kinetic equations become

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = L(f_\varepsilon) & (t,x,v) \in [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ f_\varepsilon(0,x,v) = f_{in}(x,v) & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{I.37}$$

We split the study of the behaviour of f_ε as ε goes to 0 in two steps. First, we establish a priori estimates in order to prove existence of a limit for the sequence f_ε and then we identify that limit.

I.1.3.4.1 A priori estimates

We are interested in a priori estimates that express the tendency of the system to tend towards its velocity equilibrium. The particular choice of scaling (I.35)-(I.36) actually ensures that in the limit as ε goes to 0 we will have reached the velocity-equilibrium state but not the equilibrium with respect to the position variable. This is the whole purpose of our analysis: determine the evolution of the particle density ρ at a scale where the velocities of the particle can be assumed to be distributed by a local Maxwellian.
From the dissipativity of the operator, the linearity of the kinetic equations and the regularity of the solution stated in Theorem (I.1.8), we can derive the following bounds on the density ρ_ε defined as

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,\mathrm{d}v$$

and the energy functional $\mathcal{E}_\varepsilon$, sum of kinetic energy and log-entropy:

$$\mathcal{E}_\varepsilon(t) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\frac{|v|^2}{2} + \ln f_\varepsilon\Big) f_\varepsilon\,\mathrm{d}x\,\mathrm{d}v.$$

Theorem I.1.9. *Let f_ε be a solution of of* (I.37) *in the sense of Theorem I.1.8, then we have*

i) f_ε is bounded in $L^\infty([0,+\infty), L^2_{M^{-1}}(\mathbb{R}^d\times\mathbb{R}^d))$ uniformly in ε

ii) ρ_ε is bounded in $L^\infty([0,+\infty), L^2(\mathbb{R}^d))$ uniformly in ε

ii) $\|f_\varepsilon - \rho_\varepsilon M\|_{L^2_{M^{-1}}(\mathbb{R}^d\times\mathbb{R}^d)} = O(\varepsilon)$

iv) $\mathcal{E}_\varepsilon$ is bounded in $L^\infty([0,+\infty))$ uniformly in ε.

These uniform controls yield the existence of a limit of f_ε in the following sense

Proposition I.1.10. *f_ε converges weak-$*$ in $L^\infty([0,+\infty), L^2_{M^{-1}}(\mathbb{R}^d\times\mathbb{R}^d))$ towards $\rho(t,x)M(v)$ where ρ is the weak limit of ρ_ε.*

I.1.3.4.2 Identifying the limit

In order to identify the limit, the idea of Poupaud and J.Soler in [PS00] was to follow Fourier's argument from a kinetic stand-point. They integrate the kinetic equation to recover a kinetic version of the continuity equation (I.2):

$$\partial_t \rho_\varepsilon + \frac{1}{\varepsilon} \nabla_x \cdot j_\varepsilon = 0 \tag{I.38}$$

where j_ε is the kinetic equivalent of the current density vector, defined as

$$j_\varepsilon(t,x) = \int_{\mathbb{R}^d} v f_\varepsilon(t,x,v)\,\mathrm{d}v.$$

From Fourier's law, we expect j_ε to be related to the gradient of ρ_ε, at least in the limit as ε goes to 0. Multiplying (I.37) by v, integrating and dividing by ε, they find an equation satisfied by j_ε (at least in the sense of distributions)

$$\varepsilon \partial_t j_\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v f_\varepsilon \,\mathrm{d}v = -\frac{C_L}{\varepsilon} j_\varepsilon.$$

where C_L is either 1 or d depending on the operator we consider. Moreover, they introduce a function h_ε defined as

$$h_\varepsilon(t,x,v) = \frac{1}{\varepsilon}\Big(2\nabla_v\big(\sqrt{f}\big) + v\sqrt{f}\Big)$$

with which the second term on the left-hand-side can be expressed as

$$\int_{\mathbb{R}^d} v \otimes v f_\varepsilon \,\mathrm{d}v = \varepsilon \int_{\mathbb{R}^d} h_\varepsilon \otimes v \sqrt{f_\varepsilon}\,\mathrm{d}v + \rho_\varepsilon I_d.$$

Note that h_ε satisfies

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |h_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(t).$$

Since the functional $\mathcal{E}_\varepsilon$ will be constant when the system reaches its velocity-equilibrium state, i.e. when ε goes to 0, we get

$$\int_{\mathbb{R}^d} v \otimes v f_\varepsilon \,\mathrm{d}v \to \rho I_d.$$

and as a consequence

$$\frac{C_L}{\varepsilon} j_\varepsilon \to -d\nabla_x \rho$$

which is exactly Fourier's law. In other words, we have just justified Fourier's law from a kinetic point of view by expressing how, when the system reaches its velocity equilibrium, the flux of particles through a unit of surface, j_ε, converges towards the gradient of temperature. Moreover, we see that the error between the kinetic flux and the gradient of temperature can be measured by the kinetic energy and entropy functional $\mathcal{E}_\varepsilon$.

Finally, taking the limit in the continuity equation yields the diffusion limit:

Theorem I.1.11. *The limit $\rho(t,x)$ of ρ_ε satisfies the heat equation*

$$\begin{cases} \partial_t \rho = \Delta\rho & (t,x) \in [0,+\infty) \times \mathbb{R}^d \\ \rho(0,x) = \rho_{in}(x) = \displaystyle\int_{\mathbb{R}^d} f_{in}(x,v)\,\mathrm{d}v & x \in \mathbb{R}^d. \end{cases}$$

I.2 Non-local diffusion equations

As we have seen in the previous section, one of the central results of Einstein's theory – as well as the Maxwell's and Boltzmann's work on kinetic theory and also the works of Langevin, Fokker, Planck etc – is that the mean-square-displacement of the particles scales linearly with time:

$$\langle \Delta x^2 \rangle \sim D\Delta t.$$

Despite the omnipresence of classical diffusion, which we characterise by this linearity, it is not universal. In fact, many experimental measurements have exhibited mean-square-displacements that scale as a fractional power law with time:

$$\langle \Delta x^2 \rangle \sim D\Delta t^\alpha \tag{I.39}$$

with $\alpha > 0$. This non-linearity changes drastically the diffusion phenomena and we present in this section the associated models at the microscopic, the macroscopic and the kinetic scale.
We start with a brief review of some physical experiments that illustrate non-linear mean-square-displacements. Then, we introduce the mathematical tools used to model such non-classical diffusion phenomena, both at a microscopic scale with a generalisation of Brownian motion and at a macroscopic scale with non-local diffusion operators and the associated fractional functional spaces. Finally, we introduce kinetic equations associated with the anomalous diffusion phenomena and show how we can derive macroscopic non-local diffusion equations from these kinetic models set in the whole space $\mathbb{R}^d$, generalising the results we have obtained in the classical case.

I.2.1 Motivations

We begin with the experiment of rotating annulus presented by T. Solomon, E. Weeks and H. Swinney [SWS94] in 1994 and E. Weeks, J. Urbach and H. Swinney in 1996 [WUS96]. This experiment consists in a fast rotating annulus filled with fluid that is being pumped in and out of the annulus through holes in the bottom in order to generate a turbulent flow, as illustrated in Figure I.2.

A camera on top of the annulus records the formation of turbulent eddies (small whirlpools) inside the annulus and allows the tracking of tracer particles injected into the fluid and the drawing of their orbits, as shown in Figure I.3. Looking at these orbits, we see that the trajectories of tracer particles consist of the succession

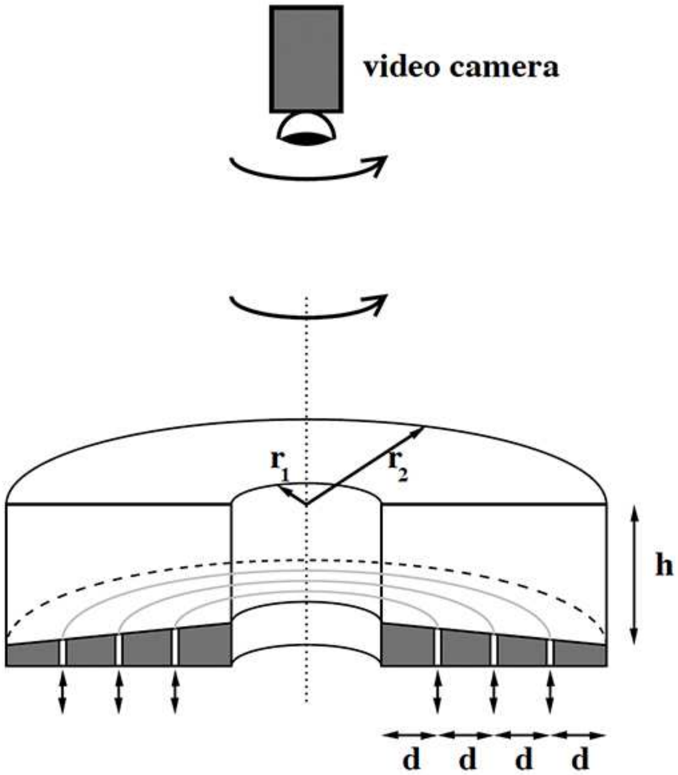


Fig. I.2 *Schematic diagram of rotating annulus from [WUS96].* $r_1 = 10.8cm$, $r_2 = 43.2cm$, $d = 8.1cm$ *and* $h = 20.3cm$ *at* r_2. *The bottom has a slope of* 0.1. *The annulus rotates rapidly is filled with fluid being pumped in and out of the annulus through small holes at the bottom.*

of *sticking times* during which they stay trapped in an eddy, and *flight times* when they travel along the edge of the annulus. As a consequence, the Brownian motion is not adapted to the modelling of these trajectories. Instead, this motivates the development of new stochastic processes adapted to these observations, as we present in section I.2.2. Weeks et al were able to show that the probability distributions of the sticking times and the flight times have power law decays $t^{-\mu}$ and $t^{-\nu}$ respectively for some μ and ν positive. Depending on the balance between μ and ν, they observe either sub-diffusion phenomena, which are slow diffusion processes with sub-linear mean-square-displacement, i.e. $\alpha < 1$ in (I.39), or super-diffusion phenomena which are fast diffusion processes characterised by a power $\alpha > 1$ in (I.39).

Another crucial example of non-local diffusion comes from the study of plasmas. Indeed, it has been recognised that the natures of transport and diffusion processes

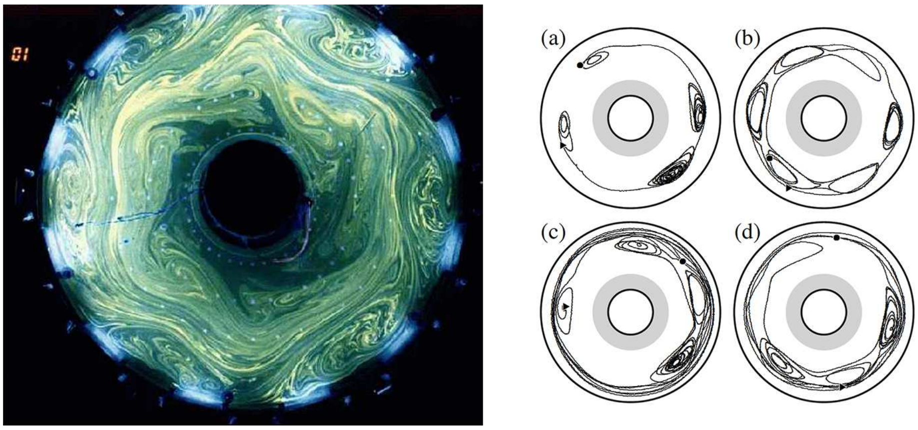


Fig. I.3 *(Left) The Formation of eddies inside the rotating annulus [VIKH08] (Right) Typical orbits of tracer particles inside the annulus [SWS94].*

that commonly occur in plasmas are dominated by turbulence with a significant non-local component. Experiments with tracer particles cannot be done with plasmas, in particular because of the extreme temperature required to maintain matter in a plasma state. In fact, experiments with plasma are very challenging and have been the subject of many works in the community of plasma physics since the beginning of the development of confinement devices in the 50s and 60s using magnetic confinement as for instance in Tokamaks and Stellarators, or inertial confinement like the NIF whose purpose is to heat a small amount of hydrogen using laser-based inertia in order to reach the plasma state. The theoretical study of turbulent plasma started in the 50s as well but since the community mainly focused on an empirical, experimental and computational approach to the problem, the theoretical framework stayed in its early stages for a few decades, until research on the systematic and mathematical justifiable modelling of turbulent plasma began again in the early 2000s as presented in Krommes' remarkable review on the subject [Kro02] from 2002.

The initial difficulty if one wants to observe the non-local phenomena occurring in plasmas, is to identify an observable quantity that characterises these phenomena. The first answer to this problem comes from the work of Mandelbrot in 1965, although he was concerned with a rather different field: Hydrology. A few years earlier, in 1956, British hydrologist Hurst observed, after decades of measurements of yearly flows of the Nile in Egypt [Hur52] that if you consider the range $R(t)$ defined as

$$R(T) = \max_{t_0 \leq t \leq t_0+T} \big[X(t) - X(t_0)\big] - \min_{t_0 \leq t \leq t_0+T} \big[X(t) - X(t_0)\big]$$

where $X(t)$ is the river's level, and the standard deviation S given by

$$S = \sqrt{\langle [X]^2 \rangle - \langle [X] \rangle^2}$$

where $\langle \cdot \rangle$ is the mean value, then the mean value of the ratio $R(T)/S$ grows as a power law:

$$\left\langle \frac{R(T)}{S} \right\rangle = CT^H$$

for some constant H which, according to Hurst's computation, is around $3/4$. Furthermore, Hurst goes on in [Hur56] to show that the same power-law growth can be observed for the river's discharges and runoffs, as well rainfalls, temperatures, pressures and annual growths of tree rings, always with a constant H which varies between $1/2$ and 1 and is usually around $3/4$. In his seminal work [Man65], Mandelbrot established a relation between Hurst's exponent H and the self-similar property of the Brownian motion. More precisely, Mandelbrot explains that if we consider the years to be independent of each other and $X(t)$ to be a Wiener process, then, as was proven by Feller in 1951 [Fel51] we have

$$\left\langle \frac{R(T)}{S} \right\rangle = CT^{1/2}$$

which is a consequence of the self-similar property of the Brownian motion, i.e. if $\{X(t), t \in \mathbb{R}\}$ is a Wiener process then for all $a > 0$

$$\{X(at), t \in \mathbb{R}\} \overset{d}{=} \{a^{1/2} X(t), t \in \mathbb{R}\}$$

where $\overset{d}{=}$ means the two process have the same finite-dimensional distributions. This fundamental property of the Brownian motion, which results from its link with the Normal distribution and expresses at the microscopic scale the scaling invariance of the heat equation (I.7), can be recovered, for instance, through the autocovariance identity of the Brownian motion: $\mathbb{E}[X(t_1)X(t_2)] = \min(t_1, t_2)$ which yields

$$\mathbb{E}\big[X(at_1)X(at_2)\big] = \min(at_1, at_2) = a \min(t_1, t_2) = \mathbb{E}\Big[\big(a^{1/2} X(t_1)\big)\big(a^{1/2} X(t_2)\big)\Big].$$

Mandelbrot arrives to the conclusion that, although the Brownian motion cannot be used to model the quantity Hurst studied, we should be able to describe them by a generalisation of the motion into a stochastic process $\{X(t), t \in \mathbb{R}\}$ with mean value zero and such that for some constant $H > 0$:

$$\{X(at), t \in \mathbb{R}\} \stackrel{d}{=} \{a^H X(t), t \in \mathbb{R}\}. \tag{I.40}$$

We will present in the next section two of the most celebrated generalisations of the Wiener process that satisfy this same **self-similar property**: the fractional Brownian motion and the Lévy flights. These generalisations are widely used in the microscopic description of non-local diffusion processes and, moreover, the Hurst exponent H answers the question we asked above: it is a measurable quantity that characterises the non-local nature of transport and diffusion in plasmas. There have been many experiments concerned with determining Hurst's exponent, often focusing on the edge fluctuation of plasma under magnetic confinement, and we refer the reader e.g. to [CHS+96] [CVMP+98] or the review paper [Car97] as well as [WSL+01], [NGNR04], [SGL+04] and [SVV04] and references within for further experimental results.

Non-local phenomena arise in many other fields and although we will not present all of them for obvious reasons, here are some references for the interested reader:

- Sub-diffusive phenomena: charge carrier transport in amorphous semiconductors [GSG+96], nuclear magnetic resonance diffusometry in percolative [KMK97] and porous systems [Kim97], rouse or reptation dynamics in polymetric systems [FKB+99], transport on fractal geometries [HMTW85] [PBHR97], the diffusion of a scalar tracer in an array of convection rolls [YPP89], the dynamics of a bead in a polymetric network [AMY+96] [BS99].

- Super-diffusive phenomena: collective slip diffusion on solid surfaces [LL99], layered velocity fields [MDM80] [ZKB91], Richardson turbulent diffusion [Ric26] [SWK87], bulk-surface exchange controlled dynamics in porous gasses [SKS95], transport in micelle systems and in heterogeneous rocks [OBLU90], quantum optics [SSY99], single molecule spectroscopy [BS99], bacteria motion [Nos83] and also in the flight of an albatross [VAB+96]

I.2.2 Microscopic description: Lévy flights

There are several ways to generalise the Wiener process in order to build a process which is self-similar with index $H > 0$. The most celebrated generalisations, the fractional Brownian motion and the Lévy flights, can be deduced from the Wiener process by removing assumptions from Definition I.1.2.

The fractional Brownian motion, introduce by Kolmogoroff [Kol93] and studied by Mandelbrot in [Man65], satisfies assumptions *i), ii)* and *iv)* but not the independent increments assumption *iii)* which will be weakened. This allows us to chose an auto-covariance function, i.e. an expression for $\mathrm{Cov}(W_{t_1}, W_{t_2}) = \mathbb{E}[W_{t_1} W_{t_2}]$, through which we ensure that (I.40) is satisfied:

Definition I.2.1. *A **fractional Brownian motion** of self-similarity index $H \in (0, 1]$ is a stochastic process W_t who satisfies:*

i) $W_0 = 0$ *almost surely*

ii) W_t *has continuous paths, i.e. it is almost surely continuous with respect to* t

iii) W_t *has stationary increment:* $W_{t+s} - W_t \sim W_s - W_0$ *, mean value* 0 *and auto-covariance function:*

$$Cov(W_{t_1}, W_{t_2}) = \frac{1}{2}\Big(|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\Big)$$

iv) W_t *has Gaussian increments: for all* $t, s \geq 0$, $W_{t+s} - W_t$ *is normally distributed with mean* 0 *and variance* s

$$W_{t+s} - W_t \sim \mathcal{N}(0, s)$$

where $\mathcal{N}(\mu, \sigma^2)$ *denotes the normal distribution with expected value* μ *and variance* σ^2.

This process has been widely studied and used for instance in the context of vortices structures in turbulent fluid [Fla02] [FG02] [Cho13], and stochastic finance [Che01] [EVDH03] [Hu05]. As we can see in Figure I.4, the fractional Brownian motion allows for a more erratic motion, hence the self-similarity with $H > 0$ and the usefulness of this motion for Mandelbrot's modelling of weather related phenomena.

However, the fractional Brownian motion does not describe an alternation between *sticking times* and *flight times* as we observed in the trajectories of particles inside a turbulent flow earlier. The Lévy flights will be more adapted for that purpose. To construct those, we will keep assumptions *i)* and *iii)* of the Wiener process, however we forgo the Gaussian increments assumption in order to allow long flights, and we weaken the continuity assumption:

Definition I.2.2. ***Lévy flights**, also called **symmetric** α**-stable Lévy processes**, are stochastic process* L_t *satisfying, for index* $\alpha \in (0, 2)$*:*

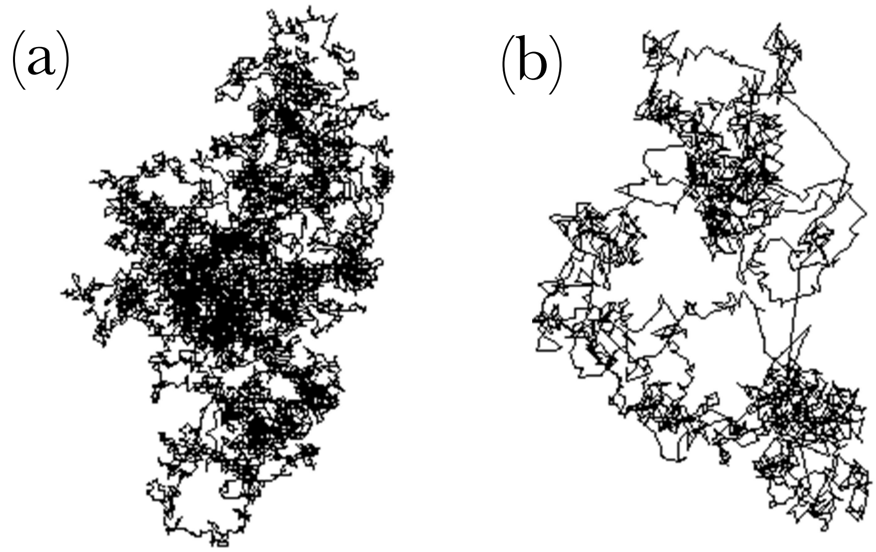


Fig. I.4 *(a) Classical Brownian motion, 10000 steps (b) Fractional Brownian motion, 10000 steps*

i) $L_0 = 0$ *almost surely*

ii) L_t *is càdlàg with respect to* t*, i.e. it is right-continuous and has left limits*

iii) L_t *has independent increments: for all* $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$*, the random variables* $L_{t_1} - L_{s_1}$ *and* $L_{t_2} - L_{s_2}$ *are independent.*

iv) L_t *has stationary symmetric Lévy increments: for all* $t, s \geq 0$*,* $L_{t+s} - L_t$ *is Lévy distributed*

$$L_{t+s} - L_t \sim \mathcal{L}_\alpha(s^{1/\alpha}, 0, 0)$$

where $\mathcal{L}_\alpha(\sigma, \beta, \mu)$ *denotes the stable Lévy distribution with index* α*, scale* σ*, skewness* β *and shift* μ*.*

We refer to [ST94] for a complete definition and analysis of the Lévy distribution $\mathcal{L}_\alpha(s^{1/\alpha}, 0, 0)$, we just note here that the associated probability density function $\lambda(x)$ decays as a polynomial:

$$\lambda(x) \underset{|x| \gg 1}{\sim} \frac{1}{|x|^{d+\alpha}}.$$

The Lévy flights are self-similar of index $H = 1/\alpha \in (1/2, +\infty)$ and we see in Figure I.5 that they describe precisely what we wanted: an succession of *sticking times* and *flight times.* We can also observe in Figure I.5 the self-similar property of the process by noticing that the unit of distance is much smaller in figure *d)* than in figure *c)* since there are 10 times as many steps in figure *d)*, which can be interpreted as a longer time interval, and yet the long flights still appear to have the same order of length in both picture, illustrating the invariance of the process under rescaling i.e. its self-similarity.

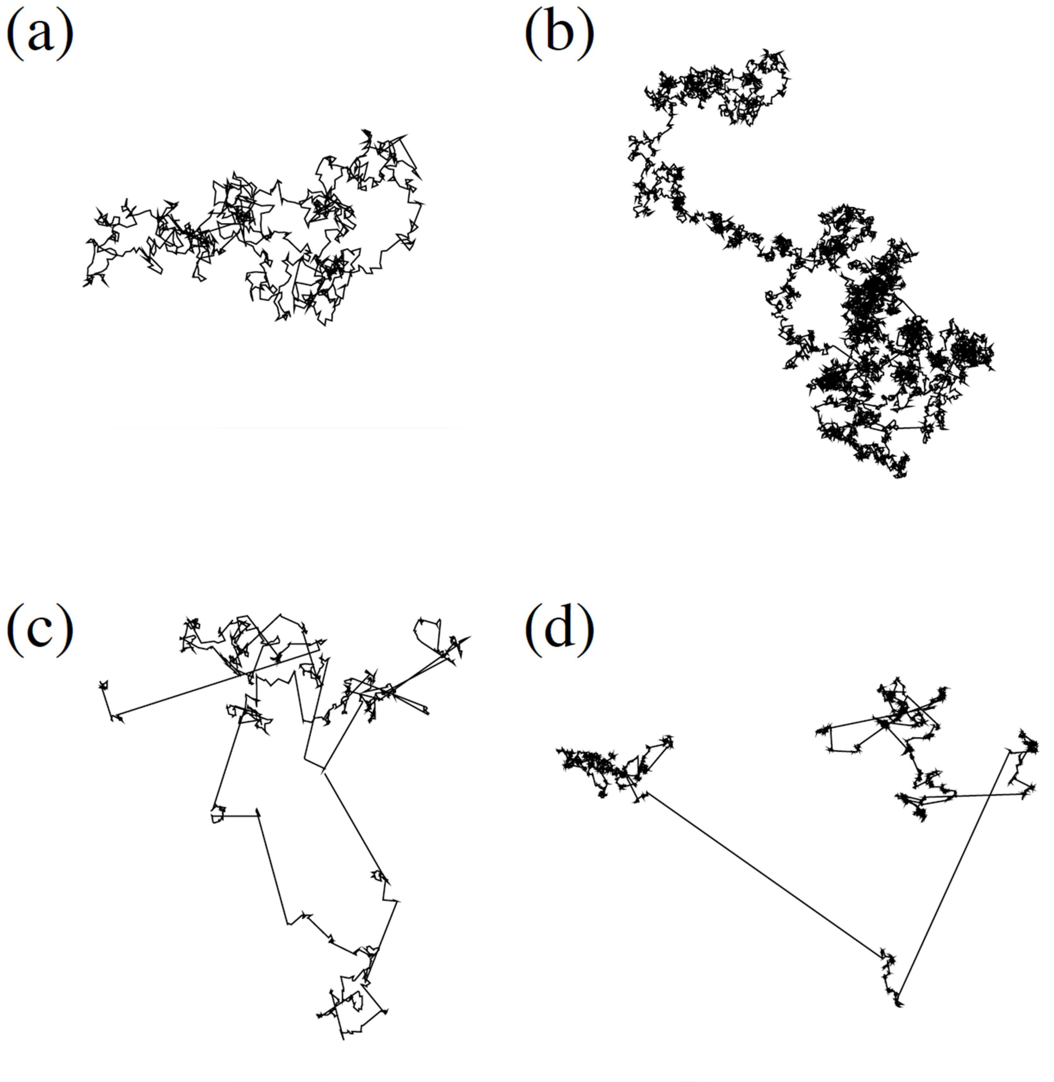


Fig. I.5 *(a) Brownian motion 1000 steps (b) Brownian motion 10000 steps (c) Lévy flight 1000 steps (d) Lévy fligth 10000 steps*

Since Lévy flights are a particular case of Lévy processes, let us note that they are infinitely divisible processes in the sense that for any t and any integer $n \geq 1$, L_t can be expressed as the sum of n independent identically distribution random variables. This is a consequence of having independent stationary increments: for $n \geq 1$ we can write explicitly:

$$L_t = L_{t/n} + \left[L_{2t/n} - L_{t/n}\right] + \cdots + \left[L_{nt/n} - L_{(n-1)t/n}\right]$$

where, thanks to *iii)* and *iv)*, the $[L_{kt/n} - L_{(k-1)t/n}]$ are independent identically distributed random variables. As a result, we can express its characteristic function $\phi(t,k)$ by the Lévy-Khintchine formulation (see e.g. [DSU08]) and, given the properties of this particular process, we actually have

$$\phi(t,k) = e^{-t|k|^{\alpha}}$$

which will be of significant importance for the macroscopic description of non-local diffusion.

I.2.3 Macroscopic description: the fractional heat equation

One way to derive the macroscopic equation for the density $\rho(t,x)$ of a cloud of particles undergoing Lévy flights is to generalise Einstein's integral conservation relation (I.11) to a general Lévy process L_t as:

$$\rho(t+\Delta t, x) - \rho(t,x) = \mathbb{E}\big(\rho(t, x + L_{\Delta t}) - \rho(t,x)\big)$$

which still expresses the idea that the evolution of the particle density ρ can be derived from the average displacement of the particles. The derivation of the macroscopic equation is equivalent to identifying the limit operator A:

$$A = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\big(\rho(t, x + L_{\Delta t}) - \rho(t,x)\big). \tag{I.41}$$

This is the infinitesimal generator of the semigroup associated with the process L_t. Namely, if we write T_t for the semigroup associated with L_t, defined as

$$T_t \rho(t,x) = \mathbb{E}\big(\rho(t, x + L_t)\big)$$

then the operator A can be equivalently defined as the infinitesimal generator of T_t

$$A\rho = \lim_{t \to 0} \frac{(T_t - 1)\rho}{t}.$$

Moreover, the characteristic function of the process L_t is related to the semigroup T_t via the Fourier transform (see e.g. [App09, Theorem 3.3.3]):

$$T_t\rho(t,x) = \mathcal{F}^{-1}\big(\phi(t,k)\hat{\rho}(t,k)\big).$$

In the case of Lévy flights $\phi(t,k) = e^{-t|k|^\alpha}$ hence

$$\mathcal{F}\big(T_t\rho(t,\cdot)\big)(k) = e^{-t|k|^\alpha}\hat{\rho}(t,k)$$

which yields

$$\mathcal{F}\big(A\rho(t,\cdot)\big)(k) = \lim_{t\to 0}\frac{e^{-t|k|^\alpha}-1}{t}\hat{\rho}(t,k) = -|k|^\alpha\hat{\rho}(t,k).$$

The resulting diffusion equation, in Fourier variable, reads

$$\partial_t\hat{\rho}(t,k) = -|k|^\alpha\hat{\rho}(t,k). \tag{I.42}$$

This is the **fractional heat equation in Fourier variables** [ARMAG00], [VTPV11], [CHS12], [BC16]. It belongs to a wide class of PDE called **non-local diffusion equations** which model a variety of non-classical diffusion phenomena, including those we are considering in this section.

I.2.3.0.1 Fractional Sobolev spaces and the fractional Laplace operator

In order to make sense of this equation in non-Fourier variable and define the fractional Laplacian, the operator whose Fourier transform is $(-A)$, we take a functional analysis approach, in the spirit of [DPV12], and introduce the fractional Sobolev space in Fourier variable:

Definition I.2.3. *In Fourier variable, the* ***fractional Sobolev space*** *of order $s \in (0,1)$: $\hat{H}^s(\mathbb{R}^d)$, is the Hilbert space defined as*

$$\hat{H}^s(\mathbb{R}^d) = \Big\{u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d}(1+|\xi|^{2s})|\hat{u}(\xi)|^2\,\mathrm{d}\xi < \infty\Big\}. \tag{I.43}$$

Note that, as is common in this framework, we have adopted the notation s for the fractional order, which is related to the previous α of the Lévy flights as

$$s = \frac{\alpha}{2}$$

and we will keep this notation here on in.
The non-Fourier version of this functional space is defined by the following proposition from [DPV12, Section 3].

Proposition I.2.1. *Consider $s \in (0,1)$. The Hilbert space $\hat{H}^s(\mathbb{R}^d)$ coincides with the **fractional Sobolev space** $H^s(\mathbb{R}^d)$ defined as*

$$H^s(\mathbb{R}^d) = \Big\{ u \in L^2(\mathbb{R}^d) : \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y < \infty \Big\}. \tag{I.44}$$

*In particular, if we define the **Gagliardo semi-norm** $[u]_{H^s(\mathbb{R}^d)}$ as*

$$[u]_{H^s(\mathbb{R}^d)} = \Bigg(\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y \Bigg)^{1/2} \tag{I.45}$$

then we have

$$\int\limits_{\mathbb{R}^d} |\xi|^{2s} \big|\hat{u}(\xi)\big|^2 \,\mathrm{d}\xi = \frac{1}{2} c_{d,s} [u]^2_{H^s(\mathbb{R}^d)} \tag{I.46}$$

where the constant $c_{d,s}$ is given by

$$c_{d,s} = \Bigg(\int\limits_{\mathbb{R}^d} \frac{1 - \cos(z_1)}{|z|^{d+2s}} \,\mathrm{d}z \Bigg)^{-1} \tag{I.47}$$

with z_1 the first coordinate of $z \in \mathbb{R}^d$.

This proposition expresses the link between a multiplication by $|\xi|^{2s}$ in Fourier variable and the singular kernel $1/|x-y|^{d+2s}$. Let us give some details about this relation which is crucial to understand non-local diffusion. We first notice, see [DPV12] for details, that the constant $c_{d,s}$ satisfies for all $\xi \in \mathbb{R}^d$:

$$(c_{d,s})^{-1} |\xi|^{2s} = \int\limits_{\mathbb{R}^d} \frac{1 - \cos(\xi \cdot z)}{|z|^{d+2s}} \,\mathrm{d}z.$$

Using this relation, we can prove the equivalence between the semi-norms (I.46) and the equivalence of the functional spaces naturally follows. For $u \in \hat{H}^s(\mathbb{R}^d)$ we write

$$\begin{aligned}\int\limits_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2\,\mathrm{d}\xi &= c_{d,s} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1-\cos(\xi\cdot z)}{|z|^{d+2s}}|\hat{u}(\xi)|^2\,\mathrm{d}z\,\mathrm{d}\xi \\ &= \frac{c_{d,s}}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|e^{i\xi\cdot z}-1|^2}{|z|^{d+2s}}|\hat{u}(\xi)|^2\,\mathrm{d}z\,\mathrm{d}\xi\end{aligned}$$

and the key step is to recognise the Fourier transform of a translation operator in the integrand on the right-hand-side. As a consequence, we have

$$\begin{aligned}\int\limits_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2\,\mathrm{d}\xi &= \frac{c_{d,s}}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big|\mathcal{F}(u(z+\cdot))(\xi)-\hat{u}(\xi)\big|^2}{|z|^{d+2s}}\,\mathrm{d}z\,\mathrm{d}\xi \\ &= \frac{c_{d,s}}{2}\int\limits_{\mathbb{R}^d} \left\|\mathcal{F}\Big(\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{d+2s}{2}}}\Big)\right\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}z\end{aligned}$$

and with the Plancherel formula this is

$$\begin{aligned}\int\limits_{\mathbb{R}^d} |\xi|^{2s}|\hat{u}(\xi)|^2\,\mathrm{d}\xi &= \frac{c_{d,s}}{2}\int\limits_{\mathbb{R}^d} \left\|\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{d+2s}{2}}}\right\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}z \\ &= \frac{c_{d,s}}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

with the change of variables $y = z + x$.
This characterisation of the fractional Sobolev spaces paves the way for the integral definition of the fractional Laplacian:

Definition I.2.4. *For a function $u \in \mathcal{S}(\mathbb{R}^d)$, the* ***fractional Laplace operator*** $\big(-\Delta\big)^s$ *is defined as*

$$\big(-\Delta\big)^s u(x) = c_{d,s} P.V.\int\limits_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y \tag{I.48}$$

where P.V. denote the Cauchy principal value. It is the inverse Fourier transform of the multiplication by $|\xi|^{2s}$:

$$\big(-\Delta\big)^s u(x) = \mathcal{F}^{-1}\Big(|\xi|^{2s}\hat{u}(\xi)\Big). \tag{I.49}$$

The kernel $1/|x-y|^{d+2s}$ is singular, hence the need for the principal value which can be defined in this situation as

$$P.V.\int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y = \lim_{\varepsilon\to 0} \int_{|y-x|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,\mathrm{d}y$$

and we see that in order for this integral to make sense we need some regularity on u and that is why we have given the definition of $\big(-\Delta\big)^s$ as an operator from the Schwartz functional space – i.e. the space of rapidly decaying smooth functions, where this regularity requirement is obviously satisfied – into $L^2(\mathbb{R}^d)$. However, we can broaden the domain of definition of $\big(-\Delta\big)^s$ by expressing its link with the fractional Sobolev spaces:

Proposition I.2.2. *If u is in $H^s(\mathbb{R}^d)$ then*

$$[u]^2_{H^s(\mathbb{R}^d)} = 2c_{d,s}^{-1}\big\|\big(-\Delta\big)^{s/2}u\big\|^2_{L^2(\mathbb{R}^d)}. \tag{I.50}$$

As a consequence, the fractional Laplacian $\big(-\Delta\big)^s$ can naturally be defined as an operator from $H^s(\mathbb{R}^d)$ into its dual space $H^{-s}(\mathbb{R}^d)$. Note that the integral definition of $\big(-\Delta\big)^s$ emphasise the **non-local nature** of the operator since in order to determine its action on a function u evaluated at a point $x\in\mathbb{R}^d$, we integrate over the whole space, hence the behaviour of u far away from x can influence the action of $\big(-\Delta\big)^s$ at x: the operator is non-local.
We have just given four equivalent definitions of the fractional Laplacian:

- as an integro-differential operator of fractional order (I.48),
- as the inverse Fourier transform of a multiplication by $|\xi|^{2s}$ (I.49),
- as the infinitesimal generator of a symmetric $2s$-stable Lévy process (I.41),
- as an operator that sends the Hilbert space $H^s(\mathbb{R}^d)$ into its dual space $H^{-s}(\mathbb{R}^d)$ (I.50).

We can also define $\big(-\Delta\big)^s$ as a fractional power of the Laplace operator in the context of functional calculus of sectorial operators, see [Hen81], or a Dirichlet-to-Neumann operator for an appropriate family of PDE on a half-space [CS07], or via its relation with Riesz potentials [CP16].
Fractional differential operators have been studied by scientists since the beginning of differential calculus. Indeed, as soon as Leibniz and Newton founded differential calculus in the XVIIth century, de l'Hôpital wondered what would happen if the differential

order was fractional which led some of the most celebrated mathematicians – including Euler, Laplace, Lacroix, Abel, Fourier, Liouville, Riemann, Laurent, Hadamard, Heavyside, Riesz and many others – to define various generalisations of classical derivatives, study the physical and mathematical relevance of these operators and see if the definitions are compatible with each other which, more often than not, was not the case, as presented in [CT14].

The multitude of equivalent definitions of the fractional Laplacian illustrates the fact that this operator arises naturally in many different fields of mathematics and, as a consequence, it is not surprising that this operator has received a lot of attention from the community, especially in recent years as the mathematical understanding of confined plasma and turbulent fluid improves and the engineering challenges involved in the confinement of plasma are closer and closer to their resolution.

The fractional Laplacian has many very interesting and useful properties. Although we will not state all of them we refer to [BV16] and [Poz16] for a full review on the subject.

As a linear operator on $H^s(\mathbb{R}^d)$, the fractional Laplacian is symmetric: for u and v in $H^s(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u\big(-\Delta\big)^s v \,\mathrm{d}x = \int_{\mathbb{R}^d} v\big(-\Delta\big)^s u \,\mathrm{d}x. \tag{I.51}$$

It actually realises the natural scalar product in the Hilbert space $H^s(\mathbb{R}^d)$ as illustrated by (I.50). Moreover, it commutes with derivatives of integer order and with other fractional Laplacians of any order. Futher, if we look at it as the infinitesimal generator of a $2s$-stable Lévy process, then it comes as no surprise that it is $2s$-homogeneous in the sense that for any $\lambda \in \mathbb{R}$ then

$$\big(-\Delta\big)^s\big[u(\lambda x)\big] = \lambda^{2s}\big(-\Delta\big)^s[u](\lambda x) \tag{I.52}$$

which expresses the self-similar property (I.40) of the underlying Lévy process.

Finally, let us mention that we can generalise the Poincaré inequality to a fractional inequality using the fractional Laplacian, although the resulting inequalities are of a very different nature since $\big(-\Delta\big)^s$ is a non-local operator. The generalisation can be done in the same space where the classical Poincaré inequality holds, i.e. an exponentially weighted L^2 space, see [MRS11]. However, in the context of fractional kinetic equations which we are about to present, it is more natural to look for a Poincaré inequality in a L^2 space with a polynomial weight. Such a generalisation was done in 2008 by I. Gentil and C. Imbert [GI08]. They took advantage of the relation between the infinitesimal generator of a Lévy process and the measure associated with the semigroup it generates in order to prove a modified logarithmic Sobolev inequality. In the case of the fractional Laplacian, the measure μ associated with the semigroup is explicitly $\mu(\,\mathrm{d}x) = F(x)\,\mathrm{d}x$ with

$$\hat{F}(\xi) = Ce^{-\frac{|\xi|^{2s}}{2s}} \tag{I.53}$$

where C is a normalising constant. The **modified logarithmic Sobolev inequality** reads, for the fractional Laplacian

Theorem I.2.3. *For all smooth positive functions u,*

$$\int_{\mathbb{R}^d} u^2 F(x)\,\mathrm{d}x - \Big(\int_{\mathbb{R}^d} uF(x)\,\mathrm{d}x\Big)^2 \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{d+2s}} F(x)\,\mathrm{d}x\,\mathrm{d}y$$

A proof of this particular case of I. Gentil and C. Imbert's result can be found in Chapter II.

Remark I.2.4. *The fractional Laplacian is an example of a wider class of non-local diffusion operators of the form*

$$Au(x) = \int_{\mathbb{R}^d} \big(u(x)-u(y)\big)K(x,y)\,\mathrm{d}y$$

for some singular kernel K, which are actually the object of study of I. Gentil and C. Imbert in [GI08]. These non-local operators can also arise in the modelling of plasmas of turbulent fluids when one considers more general Lévy processes in the microscopic scale, instead of the particular case of the Lévy flights that we presented.

I.2.3.0.2 The fractional heat equation

We can now write the **fractional heat equation** in physical variables with initial condition ρ_{in}:

$$\begin{cases} \partial_t \rho + \big(-\Delta\big)^s \rho = 0 & (t,x) \in [0,+\infty) \times \mathbb{R}^d \\ \rho(0,x) = \rho_{in}(x) & x \in \mathbb{R}^d. \end{cases} \tag{I.54}$$

We have seen that we can derive this equation from a Lévy flight model for the motion of particles. Note, however, that we did not derive this non-local equation from a generalisation of Fourier's argument. To understand why, recall that the basic idea behind Fourier's law is that in order to exit a ball B a particle must interact with its boundary, hence we can derive the evolution of the particle density in B from the interaction between the particles and the boundary ∂B as expressed in equation (I.1). In the non-local framework, there seems to be a conceptual incompatibility between the non-local behaviour of the process and the localised surface ∂B. As a consequence, generalizing Fourier's approach to the non-local diffusion case is a challenging issue.

Nevertheless, the fractional heat equation retains some of the properties of the heat equation. For instance, since the fractional Laplacian generates a semi-group, we have a fundamental solution Φ_s, solution of the evolution equation with initial condition $\rho_{in}(x) = \delta_{x=0}$, whose Fourier transform reads

$$\hat{\Phi}_s(t,k) = Ce^{-t|k|^{2s}}$$

where C is a normalising constant. As usually, we can construct general solution of (I.54) by convolution with the fundamental solution

$$\rho(t,x) = \rho_{in} * \Phi_s(t,x).$$

Moreover, a simple energy bound shows that this is the unique weak solution of the fractional heat equation in $H^s(\mathbb{R}^d)$: multiplying by ρ and integrating we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \rho^2 \,\mathrm{d}x + [\rho]^2_{H^s(\mathbb{R}^d)} = 0.$$

Since the equation is linear the difference between two weak solutions satisfies (I.54) with $\rho_{in} \equiv 0$ and the energy bound ensure that this solution stays null for all times if it is in $H^s(\mathbb{R}^d)$.

I.2.4 Kinetic equations with heavy tailed equilibrium

We will consider two kinetic descriptions of the non-local diffusion processes, generalisations of the Vlasov-linear relaxation equation (I.23) and the Vlasov-Fokker-Planck equation (I.30).

One of the most crucial differences between classical and non-local diffusion is the velocity equilibrium distribution. This distribution is a local Maxwellian distribution in the classical case. In the non-local case however, as a consequence of the high energy levels and the long flights in the microscopic motion, their is much higher concentration of high-velocity particles and the equilibrium is **heavy tailed** in the sense that it decays as a polynomial for high velocities instead of the exponential decay of the Maxwellian. If we denote by $F(v)$ this normalised equilibrium, it satisfies

$$F(v) \underset{|v|\gg 1}{\sim} \frac{1}{|v|^{d+2s}}, \qquad \int_{\mathbb{R}^d} F(v)\,\mathrm{d}v = 1. \tag{I.55}$$

The generalisation of the Vlasov-linear relaxation equation follows immediately, we just replace the equilibrium in the collision operator by this heavy-tailed F and the kinetic equation becomes:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \rho F - f & (t,x,v)\in[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, \\ f(0,x,v) = f_{in}(x,v) & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d \end{cases} \tag{I.56}$$

where $\rho(t,x) = \int_{\mathbb{R}^d} f\,\mathrm{d}v$.

To generalise the Fokker-Planck operator we introduce the **Langevin equation with a Lévy white noise**:

$$\begin{cases} \dot{x} = v(t) \\ \dot{v} = -v(t) + L_t^{2s} \end{cases} \tag{I.57}$$

where L_t^{2s} is a symmetric $2s$-stable Lévy process. Since the infinitesimal generator of this process is the fractional Laplacian, the resulting kinetic equation is the **fractional-Vlasov-Fokker-Planck equation**:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(vf) - \big(-\Delta_v\big)^s f & (t,x,v)\in[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & (x,v)\in\mathbb{R}^d\times\mathbb{R}^d. \end{cases} \tag{I.58}$$

Taking the Fourier transform in velocity of the fractional Fokker-Planck operator, the right-hand-side above, it is simple to solve for its equilibrium and recover the distribution F, defined in Fourier variables in (I.53), which indeed satisfies (I.55).

As we did in the classical case, we are interested in the diffusion limit of these equations. Introducing the Knudsen number ε and a scaling adapted to (I.40) or (I.52), namely

$$t' = \varepsilon^{2s} t, \qquad x' = \varepsilon x$$

the resulting rescaled equations take the form

$$\begin{cases} \varepsilon^{2s}\partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = L(f_\varepsilon) & (t,x,v) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ f_\varepsilon(0,x,v) = f_{in}(x,v) & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{I.59}$$

where L is either one of the previous linear operators. Since the scaling differs from the classical case, we call **anomalous diffusion limit** the study of the behaviour of f_ε when ε goes to 0. We will present separately the case of the Vlasov-linear relaxation equation with heavy tailed equilibrium in section I.2.4.1 and for the fractional Vlasov-Fokker-Planck equation in section I.2.4.2. Before that, however, let us notice that the operators we consider both satisfy the dissipativity condition defined in Proposition I.1.7 where the local Maxwellian M should be replaced by the heavy-tailed equilibrium F. The proof of this dissipativity is very similar to the classical case for the heavy-tailed relaxation operator and varies a little for the fractional Fokker-Planck and requires using the modified logarithmic Sobolev inequality of Theorem I.2.3. The dissipativity of the operators ensures the system will converge to a state of velocity-equilibrium as ε goes to 0 and eventually leads to the following a priori convergence result, common to both cases .

Proposition I.2.5. *Consider f_{in} in $L^2_{F^{-1}}(\mathbb{R}^d \times \mathbb{R}^d)$ and the weak solution f_ε of* (I.59) *in $L^\infty(0,T; L^2_{F^{-1}}(\mathbb{R}^d \times \mathbb{R}^d))$ for some time $T > 0$. Then*

$$f_\varepsilon \rightharpoonup \rho(t,x)F(v) \;\; \textit{weak-}* \;\; \textit{in } L^\infty(0,T; L^2_{F^{-1}}(\mathbb{R}^d \times \mathbb{R}^d))$$

where ρ is the weak limit of $\rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, \mathrm{d}v$.

I.2.4.1 Anomalous diffusion limit of a Vlasov-linear relaxation equation

The anomalous diffusion limit of the Vlasov-linear relaxation equation (I.56) was first derived in 2008 by A. Mellet, S. Mischler and C. Mouhot in [MMM11] through a Laplace-Fourier transform of the equation with respect to the time and space variables. Their proof differs significantly from the classical case, in particular because the Fick law (or Fourier law) fails so we cannot hope to derive the anomalous diffusion limit by means of the current density. Instead, they approach consists in taking the Laplace-Fourier transform of the equation which reads, with p the Laplace variable associated with t, and k the Fourier variable of x:

$$\varepsilon^{2s} p \hat{f}_\varepsilon - \varepsilon^{2s} \hat{f}_{in} + \varepsilon i v \cdot k \hat{f}_\varepsilon = \hat{\rho}_\varepsilon F - \hat{f}_\varepsilon.$$

Factorising appropriately and integrating with respect to v they get

$$\hat{\rho}_\varepsilon = \left(\int_{\mathbb{R}^d} \frac{F(v)}{1+\varepsilon^{2s}p+\varepsilon iv\cdot k}\,\mathrm{d}v\right)\hat{\rho}_\varepsilon + \left(\int_{\mathbb{R}^d} \frac{\varepsilon^{2s}\hat{f}_{in}}{1+\varepsilon^{2s}p+\varepsilon iv\cdot k}\,\mathrm{d}v\right)$$

and they were able to identify the limit of both terms to recover the fractional heat equation. Although this method is remarkably efficient, the use of the Fourier transform in the space variable is rather restrictive and forbids to look at space dependent collision operator or, eventually, bounded domains. This led A. Mellet in 2010 to develop a moment method for this anomalous diffusion limit in [Mel10] which we present now.

Instead of taking Fourier or Laplace transforms, Mellet's method focuses on the weak formulation of the kinetic equation and consists in choosing a particular sub-class of test functions through an auxiliary problem. The idea behind this method is that since we want, in the limit as ε goes to 0, to identify $\rho(t,x)$, we need to consider in the weak formulation all test functions for t and x but we can choose how the test function depends on the velocity as long as it does not conflict with the convergence when ε tends to 0. Hence, we build an auxiliary problem through which we construct, from $\psi\in\mathcal{D}([0,T)\times\mathbb{R}^d)$, a test function $\phi_\varepsilon(t,x,v)$ which depends on the velocity variable in an appropriate way and such that $\phi_\varepsilon(t,x,v)$ tends to $\psi(t,x)$ so that we can take the limit in the weak formulation and recover the fractional heat equation on ρ.

Mellet's moment method is fundamental to all the results we present in this thesis so let us give more details on his auxiliary problem and the convergence of the weak formulation.

I.2.4.1.1 Auxiliary problem

For a test function $\psi\in\mathcal{D}([0,+\infty)\times\mathbb{R}^d)$ we construct $\phi_\varepsilon(t,x,v)$ in $L^\infty((0,+\infty)\times\mathbb{R}^d_v;L^2(\mathbb{R}^d_x))$ as a solution of

$$\phi_\varepsilon - \varepsilon v\cdot\nabla_x\phi_\varepsilon = \psi(t,x). \tag{I.60}$$

This equation is easily integrated and we have an explicit formula for ϕ_ε:

$$\phi_\varepsilon(t,x,v) = \int_0^{+\infty} e^{-z}\psi(t,x+\varepsilon vz)\,\mathrm{d}z.$$

Futher, we see that ϕ_ε is smooth, bounded in L^∞ and also:

$$\begin{aligned}|\phi_\varepsilon(t,x,v)-\psi(t,x)| &= \left|\int_0^{+\infty} e^{-z}\big[\psi(t,x+\varepsilon vz)-\psi(t,x)\big]\,\mathrm{d}z\right| \\ &\leq \varepsilon|v|\|\psi\|_{L^\infty((0,+\infty)\times\mathbb{R}^d)}\end{aligned}$$

hence

$$\phi_\varepsilon(t,x,v) \underset{\varepsilon\to 0}{\longrightarrow} \psi(t,x) \quad \text{uniformly with respect to } t \text{ and } x.$$

However, the convergence is not uniform in v but it is not an obstacle to the convergence of the weak formulation because it satisfies

Lemma I.2.6. *Consider $\psi \in \mathcal{D}([0,+\infty)\times\mathbb{R}^d)$ and ϕ_ε solution of* (I.60)*. We have*

$$\begin{cases}\displaystyle\int_{\mathbb{R}^d}\big[\phi_\varepsilon(t,x,v)-\psi(t,x)\big]F(v)\,\mathrm{d}v \underset{\varepsilon\to 0}{\longrightarrow} 0 & \textit{uniformly with respect to } t \textit{ and } x, \\ \displaystyle\int_{\mathbb{R}^d}\big[\partial_t\phi_\varepsilon(t,x,v)-\partial_t\psi(t,x)\big]F(v)\,\mathrm{d}v \underset{\varepsilon\to 0}{\longrightarrow} 0 & \textit{uniformly with respect to } t \textit{ and } x.\end{cases}$$

Furthermore,

$$\begin{cases}\|\phi_\varepsilon\|_{L^2_F((0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d)} \leq \|\psi\|_{L^2_F((0,+\infty)\times\mathbb{R}^d)}, \\ \|\partial_t\phi_\varepsilon\|_{L^2_F((0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d)} \leq \|\partial_t\psi\|_{L^2_F((0,+\infty)\times\mathbb{R}^d)}.\end{cases}$$

I.2.4.1.2 Identifying the limit

The weak formulation of (I.56) on $Q=[0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$ with test function $\phi(t,x,v)$ reads

$$\iiint_Q \Big[f_\varepsilon\Big(\varepsilon^{2s}\partial_t\phi+\varepsilon v\cdot\nabla_x\phi-\phi\Big)+\rho_\varepsilon F\phi\Big]\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \varepsilon^{2s}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_{in}\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

Taking the solution ϕ_ε of the auxiliary problem as test function, we see that we have (since F is normalised)

$$\iiint_Q \Big[f_\varepsilon \Big(\varepsilon v \cdot \nabla_x \phi_\varepsilon - \phi_\varepsilon \Big) + \rho_\varepsilon F \phi_\varepsilon \Big] \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}v$$
$$= \iint_{[0,+\infty)\times\mathbb{R}^d} \rho_\varepsilon \int_{\mathbb{R}^d} \big[\phi_\varepsilon(t,x,v) - \psi(t,x)\big] F(v) \,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}v.$$

Introducing the operator $\mathcal{L}^\varepsilon$ defined as

$$\mathcal{L}^\varepsilon(\psi) = \varepsilon^{-2s} \int_{\mathbb{R}^d} \big[\phi_\varepsilon(t,x,v) - \psi(t,x)\big] F(v) \,\mathrm{d}v \tag{I.61}$$

the weak formulation becomes

$$\iint_{[0,+\infty)\times\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f_\varepsilon \partial_t \phi_\varepsilon \,\mathrm{d}v + \rho_\varepsilon \mathcal{L}^\varepsilon(\psi) \Big) \,\mathrm{d}t \,\mathrm{d}x = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_{in}(x,v) \phi_\varepsilon(0,x,v) \,\mathrm{d}x \,\mathrm{d}v. \tag{I.62}$$

Note that this weak formulation does not identify f_ε as the solution of the heavy-tailed Vlasov-linear relaxation equation (I.56) because it is only satisfied for a particular sub-class of test functions. However, the solution of (I.56) does satisfy (I.62) for all ϕ_ε solution of (I.60) and the structure of a diffusion equation appears where $\mathcal{L}^\varepsilon$ is a kinetic approximation of a non-local diffusion operator.
The rest of the proof consists in taking the limit as ε goes to zero in this formulation. The convergence of the partial derivative with respect to time and the initial condition follows from the a priori estimates and the bounds on ϕ_ε. We focus on $\mathcal{L}^\varepsilon$. From the explicit expression of ϕ_ε we have

$$\mathcal{L}^\varepsilon(\psi) = \varepsilon^{-2s} \int_{\mathbb{R}^d} \int_0^{+\infty} e^{-z} \big[\psi(t, x + \varepsilon v z) - \psi(t,x)\big] F(v) \,\mathrm{d}z \,\mathrm{d}v.$$

The convergence of this operator towards the fractional Laplacian rests upon the relation between F and the singular kernel of $\big(-\Delta\big)^s$. Indeed, we know that $F(v) \sim \kappa_0/|v|^{d+2s}$ for large v and some $\kappa_0 > 0$, so the change of variable $w = \varepsilon v z$ yields formally

$$\begin{aligned}
\mathcal{L}^\varepsilon(\psi) &= \varepsilon^{-2s} \int_{\mathbb{R}^d} \int_0^{+\infty} e^{-z} \big[\psi(t,x+w) - \psi(t,x)\big] F\Big(\frac{w}{\varepsilon z}\Big) \frac{1}{|\varepsilon z|^d} \,\mathrm{d}z\,\mathrm{d}w \\
&\underset{\varepsilon \ll 1}{\sim} \varepsilon^{-2s} \int_{\mathbb{R}^d} \int_0^{+\infty} e^{-z} \big[\psi(t,x+w) - \psi(t,x)\big] \frac{(\varepsilon z)^{d+2s}}{|w|^{d+2s}} \frac{1}{|\varepsilon z|^d} \,\mathrm{d}z\,\mathrm{d}w \\
&\underset{\varepsilon \ll 1}{\sim} \int_{\mathbb{R}^d} \int_0^{+\infty} z^{2s} e^{-z} \frac{\psi(t,x+w) - \psi(t,x)}{|w|^{d+2s}} \,\mathrm{d}z\,\mathrm{d}w \\
&\underset{\varepsilon \to 0}{\longrightarrow} -\kappa\big(-\Delta_x\big)^s \psi(t,x)
\end{aligned}$$

where the constant κ is given by

$$\kappa = \frac{\kappa_0}{c_{d,s}} \int_0^{+\infty} z^{2s} e^{-z} \,\mathrm{d}z.$$

The rigorous proof of this limit can be done by splitting the integral over v in two: $\{|v| \geq C\} \cup \{|v| \leq C\}$ and showing that the integral over small velocity vanishes while, in the integral over large velocities, the equilibrium F converges to the singular kernel $\kappa_0/|v|^{d+2s}$. See [Mel10] for more details.
Put together, the limit of (I.62) identifies the limit ρ of $f_\varepsilon/F(v)$ as solution of

$$\iint_{[0,+\infty)\times\mathbb{R}^d} \rho\big(\partial_t \psi - \kappa\big(-\Delta_x\big)^s \psi\big) \,\mathrm{d}t\,\mathrm{d}x = \int_{\mathbb{R}^d} \rho_{in} \psi(0,x) \,\mathrm{d}x$$

for all $\psi \in \mathcal{D}([0,+\infty) \times \mathbb{R}^d)$ and the uniqueness of weak solution of the fractional heat equation in H^s ensure that ρ is this unique solution.

I.2.4.2 Anomalous diffusion limit of a fractional Vlasov-Fokker-Planck equations

The anomalous diffusion limit of the fractional Vlasov-Fokker-Planck equation (I.58) was derived in 2012 by myself, A. Mellet and K. Trivisa [CMT12] [1]. Although the limit can be obtained through a Fourier method we will focus here a moment method

[1] Erratum for [CMT12]: In the proof of [CMT12, Proposition 2.1] the Poincaré inequality of [MRS11] does not hold, one needs to use instead the modified log-Sobolev inequality of [GI08]. The results remains unchanged.

for the same reasons as before. Building on Mellet's idea, we want to construct a particular sub-class of test functions in order to isolate the diffusion phenomena in the weak formulation, creating an adapted kinetic approximation of the fractional heat equation, and then take the limit in this approximation. However, this sub-class of test functions will take a different form for the fractional Vlasov-Fokker-Planck equation because we already have an explicit non-local operator in the collision model, acting on the velocities. As a consequence, the purpose of the auxiliary problem will be to identify a relevant relation between the position and the velocity variable through which we can exhibit how the non-local phenomena in the behaviour of the velocities of particles in the microscopic scale (c.f. the Langevin equation with Lévy white noise (I.57)) results in non-local behaviour for the particle density ρ at the macroscopic scale.

I.2.4.2.1 Auxiliary problem

To build the auxiliary problem, we take advantage of the particular structure the fractional Vlasov-Fokker-Planck equation exhibits when we take its Fourier transform in position and velocity. Indeed, with Fourier variables p and ξ for x and v respectively, the rescaled equation reads

$$\varepsilon^{2s}\partial_t \hat{f}_\varepsilon + \big(\varepsilon k - \xi\big) \cdot \nabla_\xi \hat{f}_\varepsilon = -|\xi|^{2s}\hat{f}_\varepsilon$$

which is a scalar-hyperbolic equation whose characteristic lines are given by the term $(\varepsilon k - \xi) \cdot \nabla_\xi$. This motivates the following auxiliary problem to construct ϕ_ε from $\psi \in \mathcal{D}([0,+\infty) \times \mathbb{R}^d)$:

$$\begin{cases} \varepsilon v \cdot \nabla_x \phi_\varepsilon - v \cdot \nabla_v \phi_\varepsilon = 0 & (t,x,v) \in [0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d \\ \phi_\varepsilon(t,x,0) = \psi(t,x) & (t,x) \in [0,+\infty) \times \mathbb{R}^d. \end{cases} \tag{I.63}$$

In this setting, we have an explicit solution for ϕ_ε which is

$$\phi_\varepsilon(t,x,v) = \psi(t, x + \varepsilon v).$$

which is smooth in all variables. Moreover

$$\phi_\varepsilon(t,x,v) \underset{\varepsilon \to 0}{\longrightarrow} \psi(t,x) \qquad \text{in } \mathcal{D}([0,+\infty) \times \mathbb{R}^d).$$

so it will does not conflict with the convergence of the associated weak formulation.

I.2.4.2.2 Identifying the limit

The weak formulation of (I.58) on $Q = [0,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ with test function ϕ reads

$$\iiint\limits_Q f_\varepsilon\Big(\varepsilon^{2s}\partial_t\phi + \varepsilon v\cdot\nabla_x\phi - v\cdot\nabla_v\phi - \big(-\Delta_v\big)^s\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$
$$= \varepsilon^{2s}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f_{in}\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v.$$

With the test function ϕ_ε above, it becomes

$$\iiint\limits_Q f_\varepsilon\Big(\varepsilon^{2s}\partial_t\phi_\varepsilon - \big(-\Delta_v\big)^s\phi_\varepsilon\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \varepsilon^{2s}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f_{in}\phi_\varepsilon(0,x,v)\,\mathrm{d}x\,\mathrm{d}v$$

and since $\big(-\Delta\big)^s$ is $2s$-homogeneous (I.52) we have

$$\big(-\Delta_v\big)^s\phi_\varepsilon = \big(-\Delta\big)^s\big[\phi(t,x+\varepsilon v)\big] = \varepsilon^{2s}\big(-\Delta\big)^s\big[\psi(t,\cdot)\big](x+\varepsilon v).$$

Hence, the weak formulation can be written as

$$\iiint\limits_Q f_\varepsilon\Big(\partial_t\psi(t,x_\varepsilon v) - \big(-\Delta\big)^s\big[\psi(t,\cdot)\big](x+\varepsilon v).\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f_{in}\psi(0,x+\varepsilon v)\,\mathrm{d}x\,\mathrm{d}v$$

and the rest of the proof consists in taking the limit as ε goes to 0. Notice that we cannot write explicitly an operator $\mathcal{L}^\varepsilon$ independent of v as we did in the linear relaxation case to approximate the fractional Laplacian. Nevertheless, the strong convergence of ϕ_ε towards $\psi(t,x)$ ensures that

$$\iiint\limits_Q f_\varepsilon\big(-\Delta\big)^s\big[\psi(t,\cdot)\big](x+\varepsilon v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \underset{\varepsilon\to0}{\longrightarrow} \Bigg(\iint\limits_{[0,T)\times\mathbb{R}^d}\rho\big(-\Delta_x\big)^s\psi\,\mathrm{d}t\,\mathrm{d}x\Bigg)\Bigg(\int\limits_{\mathbb{R}^d}F(v)\,\mathrm{d}v\Bigg)$$

so we recover the fractional heat equation in the limit

$$\iint\limits_{[0,+\infty)\times\mathbb{R}^d}\rho\big(\partial_t\psi - \big(-\Delta_x\big)^s\psi\big)\,\mathrm{d}t\,\mathrm{d}x = \int\limits_{\mathbb{R}^d}\rho_{in}\psi(0,x)\,\mathrm{d}x$$

for all $\psi\in\mathcal{D}([0,+\infty)\times\mathbb{R}^d)$.

I.3 Confining a diffusion process

Now that we have presented the different models of local and non-local diffusion phenomena at the microscopic, the kinetic and the macroscopic scale and seen the relations between them, we turn to the main focus of this thesis: the confinement of diffusion processes.
We consider two types of confinement: "soft" confinement with an external electric field and "hard" confinement with a bounded domain. Of course, there are other ways to confine a diffusion process, for instance with a self-consistent electric field (given by a Poisson equation), or we could also consider the free boundary problem which may exhibit a confinement resulting from the balance of attracting and repulsing forces inside the fluid.
We start this section with the electric field case, introducing the problem and giving some results in the classical diffusion setting. Next, we consider bounded domains, introduce the classical macroscopic boundary conditions for the heat equation, and present some of the associated results before moving on to the kinetic boundary conditions that Maxwell introduced in the late XIXth century. Then, building from the classical diffusion limits, we show how we can recover the macroscopic boundary conditions from the kinetic ones. Finally, we investigate the confinement of non-local diffusion processes and look at this problem from both a microscopic and a macroscopic point of view, presenting some of the most recent results on that subject.

I.3.1 External electric field

Let us consider a rarefied gas, or a fluid, near equilibrium and subject to an external electric field $E(t,x)$ which derives from a electric potential Φ: $E = \nabla_x \Phi$. At the microscopic scale, the field affects the velocity of the particles and hence modifies the Langevin description (I.29) of the evolution of $(x(t), v(t))$ the position and velocity of a given particle, which becomes

$$\begin{cases} \dot{x} = v(t) \\ \dot{v} = E(t,x) - \mu v(t) + DB_t \end{cases} \tag{I.64}$$

where μ and D, the viscosity and diffusion constants, will be assumed equal to 1 from now on, and B_t is a Wiener process. We can see here that, for example, if the vector E is oriented towards the origin, then the field discourages particles from moving away from the origin, hence the term "soft" confinement.

The resulting kinetic equation is a linear Vlasov-Fokker-Planck with an electric field:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = \nabla_v \cdot (vf) + \Delta_v f & (t,x,v) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{I.65}$$

This equation can be interpreted as a perturbation of the linear Vlasov-Fokker-Planck equation (I.30) in the sense that the collision operator is perturbed by the electric field and becomes

$$L_{FP,E} f = \nabla_v \cdot \Big[(v - E(t,x)) f \Big] + \Delta_v f.$$

If the perturbation is "nice enough", namely if E is in $L^\infty((0,T) \times \mathbb{R}^d)$ – note that $E(t,x) \in \mathbb{R}^d$ so that when we say E is in some functional space $\mathcal{F}$ we mean that if we write $E(t,x) = (E_1(t,x), \dots, E_d(t,x))$ then each of the component $E_i(t,x)$ is in $\mathcal{F}$ – then it does not affect the fundamental properties of the equation and we can prove similar existence and regularity results as in the unperturbed case, as was done for instance in [BD95] or [EGM10]:

Proposition I.3.1. *Consider $T > 0$, $E \in L^\infty((0,T) \times \mathbb{R}^d)$ and $f_{in} \in L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d)$ such that*

$$f_{in} \geq 0 \text{ and } \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} (1 + |v|^2 + \ln f_{in}) f_{in} \, \mathrm{d}x \, \mathrm{d}v < \infty$$

then (I.65) *has a weak solution $f \in \mathcal{C}([0,T); L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d)$ that satisfies*

$$f \geq 0 \text{ and for all } t \in [0,T) : \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} (1 + |v|^2 + \ln f) f \, \mathrm{d}x \, \mathrm{d}v < \infty.$$

Advection-diffusion limit

Let us derive the macroscopic equation on ρ that follows from this perturbed Vlasov-Fokker-Planck equation. Since we are in the classical case, we use rescaling

$$t' = \varepsilon^2 t, \qquad x' = \varepsilon x$$

where ε is the Knudsen number (I.34). In order to investigate the limit as ε goes to 0, we need to know how E rescales with ε. Since E derives from a potential Φ, we see that

$$E(\varepsilon^2 t, \varepsilon x) = \nabla_x \big[\Phi(\varepsilon^2 t, \varepsilon x)\big] = \varepsilon \nabla_x \Phi(\varepsilon^2 t, \varepsilon x) = \varepsilon E(t', x'). \tag{I.66}$$

Hence, the rescaled kinetic equation reads

$$\begin{aligned} &\varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon + \varepsilon E \cdot \nabla_v f_\varepsilon = \nabla_v \cdot (v f_\varepsilon) + \Delta_v f_\varepsilon && \text{on } [0,T) \times \mathbb{R}^d \times \mathbb{R}^d \\ &f_\varepsilon(0,x,v) = f_{in}(x,v) && \text{on } \mathbb{R}^d \times \mathbb{R}^d. \end{aligned} \tag{I.67}$$

We can investigate the behaviour of f_ε as ε goes to 0 by adapting the method developed in Section I.1.3.4 for the case without the electric field:

Proposition I.3.2. *Consider $T > 0$ and f_{in} satisfying the assumption of Proposition I.3.1. Then the solution of the rescaled equation* (I.67) *converges towards $\rho(t,x)M(v)$ weak-$*$ in $L^\infty([0,T); L^2_{M^{-1}}(\mathbb{R}^d \times \mathbb{R}^d))$ where M is the local Maxwellian equilibrium* (I.31) *and ρ is the weak limit of $\rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \,\mathrm{d}v$.*

We also retain uniform control on the energy

$$\mathcal{E}_\varepsilon(t,x) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \Big(\frac{|v|^2}{2} + \ln f_\varepsilon\Big) f_\varepsilon \,\mathrm{d}x \,\mathrm{d}v.$$

Details on this a priori estimates can be found in [PS00] or [EGM10], and also in Chapter II of this thesis. We focus here on identifying the limit. To that end, we integrate the equation to derive the continuity equation for the density ρ_ε:

$$\partial_t \rho_\varepsilon + \frac{1}{\varepsilon} \nabla_x j_\varepsilon = 0$$

where j_ε is now the current density

$$j_\varepsilon = \int_{\mathbb{R}^d} v f_\varepsilon \,\mathrm{d}v.$$

The a priori estimates ensure that $\frac{1}{\varepsilon} j_\varepsilon$ converges, we want to identify its limit. Multiplying (I.67) by v/ε and integrating we have in the sense of distributions

$$\varepsilon \partial_t j_\varepsilon + \nabla_x \int_{\mathbb{R}^d} v \otimes v f_\varepsilon \,\mathrm{d}v - dE(t,x)\rho_\varepsilon = -\frac{d}{\varepsilon} j_\varepsilon.$$

We know that the second term on the left-hand-side will tend to $-\nabla_x \cdot (\rho I_d)$ and the bounds on ρ_ε ensure that the third term on the left-hand-side will tend to $-dE\rho$ hence

$$\frac{d}{\varepsilon} j_\varepsilon \to d\nabla_x \rho + dE(t,x)\rho$$

and together with the continuity equation we get the advection-diffusion limit of the Vlasov-Fokker-Planck equation with an external electric field:

$$\begin{cases} \partial_t \rho - \nabla_x \cdot (\nabla_x \rho + E(t,x)\rho) = 0 & (t,x) \in [0,T) \times \mathbb{R}^d \\ \rho(0,x) = \rho_{in}(x) & x \in \mathbb{R}^d \end{cases} \tag{I.68}$$

As expected, this equation models the evolution of ρ under the effects of a diffusion and an advection resulting from the electric field.

I.3.2 Bounded domains

We consider a bounded domain $\Omega \subset \mathbb{R}^d$ and consider a fluid confined in that domain. We will always assume that Ω is smooth in the sense that there exists a smooth function $\xi : \mathbb{R}^d \mapsto \mathbb{R}$ such that

$$\Omega = \left\{ x \in \mathbb{R}^d : \xi(x) < 0 \right\} \text{ and } \partial\Omega = \left\{ x \in \mathbb{R}^d : \xi(x) = 0 \right\}$$

and we also assume that $\nabla_x \xi(x) \neq 0$ for all $|\xi(x)| \ll 1$ so that we can define the outward normal vector $n(x) = \nabla_x \xi(x) / |\nabla_x \xi(x)|$ everywhere on the boundary.

I.3.2.1 Macroscopic boundary conditions for classical diffusion equations

Let us consider the heat equation in Ω:

$$\begin{cases} \partial_t \rho = \Delta \rho & (t,x) \in [0,T) \times \Omega \\ \rho(0,x) = \rho_{in}(x) & x \in \Omega. \end{cases} \tag{I.69}$$

In order to close this problem we need to describe how ρ behaves on the boundary. There are two fundamental ways to do this, either impose the value of ρ on the boundary, or the value of its normal derivative. We focus on the homogeneous conditions:

- **Homogeneous Dirichlet boundary condition**

$$\rho(t,x) = 0 \qquad (t,x) \in [0,T) \times \partial\Omega \tag{I.70}$$

- **Homogeneous Neumann boundary condition**

$$\nabla_x \rho(t,x) \cdot n(x) = 0 \qquad (t,x) \in [0,T) \times \partial\Omega. \tag{I.71}$$

We can build solutions to the initial-boundary-value problems (I.69)-(I.70) and (I.69)-(I.71) using the eigenvalues of the Laplacian and the associated orthonormal basis of $L^2(\Omega)$, see e.g. [Eva10] or [Tay11]. Note that in the Neumann case, we have conservation of mass since the boundary is reflective, as we can see easily by integrating the equation, but that is not necessarily true in the Dirichlet case. In both cases, we have uniqueness of solution, in $H_0^1(\Omega)$ for Dirichlet, and $H^1(\Omega)$ for Neumann. Indeed, a simple energy estimate shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(t,x)^2 \,\mathrm{d}x + \int_{\partial\Omega} \left|\nabla_x \rho\right|^2 \mathrm{d}x = 0.$$

Using the Poincaré inequality, it follows that the H^1-norm decreases, hence the uniqueness since the equation is linear. Moreover, the solutions satisfy a remarkable property: the maximum principle, which illustrates the diffusive effect of the equation by the fact that the maximum value of the $\rho(t,x)$ can only be attained on the boundary or by the initial value:

Proposition I.3.3. *Let ρ be a solution of* (I.69) *with either Dirichlet or Neumann boundary condition in $\mathcal{C}([0,T) \times \bar{\Omega}) \cap \mathcal{C}^2((0,T) \times \Omega)$ then*

$$\sup_{[0,T)\times\bar{\Omega}} \rho(t,x) = \max\left\{\sup_{\bar{\Omega}} \rho(0,x), \sup_{[0,T)\times\partial\Omega} \rho(t,x)\right\}$$

and we refer to [Tay11] or [Eva10] for more a more detailed analysis of these equations.

I.3.2.2 Kinetic boundary conditions

Let us consider the linear Vlasov-Fokker-Planck equation on Ω:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (vf) + \Delta f & (t,x,v) \in [0,T) \times \Omega \times \mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & (x,v) \in \Omega \times \mathbb{R}^d. \end{cases} \tag{I.72}$$

Here again, we need to specify how f behaves on the boundary to which end we introduce the oriented set:

$$\Sigma_{\pm} = \{(x,v) \in \Sigma; \pm n(x)\cdot v > 0\} \text{ with } \Sigma = \partial\Omega \times \mathbb{R}^d \tag{I.73}$$

where $n(x)$ is the outgoing normal vector and we denote by γf the trace of f on $\mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^d$. The boundary conditions then take the form of a balance between the values of the traces of f on these oriented sets $\gamma_\pm f := \mathbb{1}_{\Sigma_\pm}\gamma f$. Maxwell identified in [Max79] three fundamental interactions between the cloud of particles and the boundary which give rise to the following boundary conditions:

- The absorption boundary condition : for all $(x,v) \in \Sigma_-$

$$\gamma_- f(t,x,v) = 0 \tag{I.74}$$

- The local-in-velocity reflection operator called *specular reflection*: for all $(x,v) \in \Sigma_-$

$$\gamma_- f(t,x,v) = \gamma_+ f\big(t,x,\mathcal{R}_x(v)\big) \tag{I.75}$$

  where $\mathcal{R}_x(v) = v - 2\big(n(x)\cdot v\big)n(x)$ which is illustrated in Figure I.6.

- The non-local in velocity reflection operator called *diffusion* : for all $(x,v) \in \Sigma_-$

$$\gamma_- f(t,x,v) = M(v)\int_{\Sigma_+^x} \gamma_+ f(t,x,w)|n(x)\cdot w|\,\mathrm{d}w \tag{I.76}$$

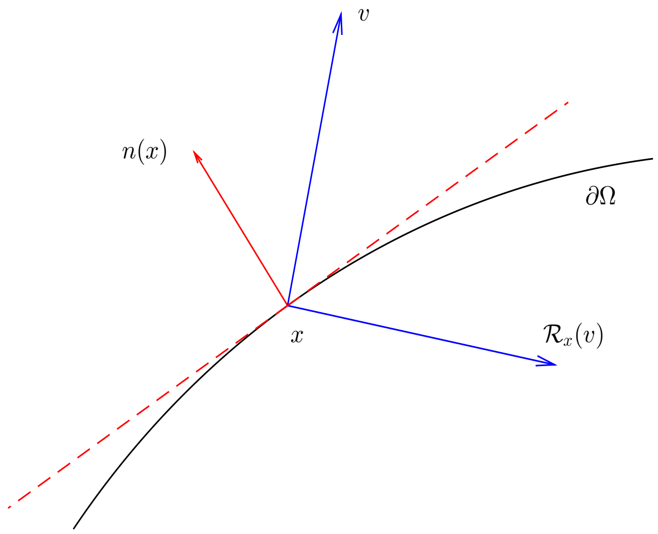
  where M is the Gaussian equilibrium (I.31) with the normalising assumption

$$\int_{\Sigma_-^x} M(w)|w\cdot n(x)|\,\mathrm{d}w = 1.$$

The first one models the absorption of the particles by the boundary, the second expresses the reflection of the particle that bounces back with a reflected velocity and the third is when the boundary diffuses back into the domain. For a reflective boundary, the most physically relevant model for the interaction would be a linear combination of specular reflection and diffusion, i.e. for some $\theta \in (0,1)$:

$$\gamma_- f(t,x,v) = \theta\gamma_+ f\big(t,x,\mathcal{R}_x(v)\big) + (1-\theta)M(v)\int_{\Sigma_+^x} \gamma_+ f(t,x,v)|n(x)\cdot v|\,\mathrm{d}v$$

Fig. I.6 Specular reflection operator



which is usually called the **Maxwell reflection boundary condition**.
The existence and regularity of solution, up to the boundary, of kinetic equation with either one of the boundary conditions has been the subject of many works such as for instance [Bar70], [Ces84], [Ces85], [CC91], [RW92], [AC93], [AM94], [CS95], [Car98] and more recently [Mis10].
We are interested in the diffusion limit of a rescaled Vlasov-Fokker-Planck equation on a bounded domain and we notice that the three boundary conditions are invariant by the classical rescaling (I.35)-(I.36) so we consider the rescaled kinetic equation

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \nabla_v \cdot (v f_\varepsilon) + \Delta f_\varepsilon & (t,x,v) \in [0,T) \times \Omega \times \mathbb{R}^d \\ f_\varepsilon(0,x,v) = f_{in}(x,v) & (x,v) \in \Omega \times \mathbb{R}^d. \end{cases} \tag{I.77}$$

with either (I.74), (I.75) or (I.76) on the boundary. The behaviour of f_ε as ε goes to 0 has been investigated in [CSV96], [BCS97] and [WLL15a] amongst others. In the reflective cases (I.75) and (I.76) we can actually see the macroscopic boundary condition arise from the kinetic ones by looking at the scalar product of the current density j_ε at a point x on the boundary and the normal vector $n(x)$. We see that for specular reflections

$$
\begin{aligned}
j_\varepsilon(t,x)\cdot n(x) &= \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon v\cdot n(x)\,\mathrm{d}v + \int\limits_{\Sigma_-^x} \gamma_- f_\varepsilon v\cdot n(x)\,\mathrm{d}v \\
&= \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon |v\cdot n(x)|\,\mathrm{d}v - \int\limits_{\Sigma_-^x} \gamma_+ f_\varepsilon(x,\mathcal{R}_x(v)) v\cdot n(x)\,\mathrm{d}v \\
&= \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon |v\cdot n(x)|\,\mathrm{d}v - \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon(x,w)\mathcal{R}_x(w)\cdot n(x)\,\mathrm{d}w \\
&= 0
\end{aligned}
$$

since $\mathcal{R}_x(w)\cdot n(x) = -w\cdot n(x)$ as a direct consequence of the definition of $\mathcal{R}_x$. Similarly for diffusive boundary conditions:

$$
\begin{aligned}
j_\varepsilon(t,x)\cdot n(x) &= \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon v\cdot n(x)\,\mathrm{d}v + \int\limits_{\Sigma_-^x} M(v) v\cdot n(x)\,\mathrm{d}v \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon w\cdot n(x)\,\mathrm{d}w \\
&= 0
\end{aligned}
$$

thanks to the normalising assumption on M. Hence, we have in both cases

$$
j_\varepsilon(t,x)\cdot n(x) = 0 \qquad \forall (t,x)\in[0,T)\times\partial\Omega.
$$

Since we know that $\frac{1}{\varepsilon} j_\varepsilon \to \nabla_x \rho$, it follows that both the specular reflection and the diffusion boundary conditions give rise to the homogeneous Neumann condition on ρ in the diffusion limit.

I.3.2.3 Boundary conditions for non-local diffusion equations

If we consider the fractional heat equation on a domain Ω

$$
\begin{cases}
\partial_t \rho - \big(-\Delta\big)^s \rho = 0 & (t,x)\in[0,T)\times\Omega \\
\rho(0,x) = \rho_{in}(x) & x\in\Omega
\end{cases}
\tag{I.78}
$$

then it is not obvious, from a PDE point of view, what kind of boundary condition we should consider in order to close this problem. Indeed, because of the non-local nature of the fractional Laplacian, a Dirichlet or a Neumann boundary condition on $\partial\Omega$ would result in an ill-posed problem since these conditions are local in space.

The problem of confining of non-local diffusion processes arose first in the field of

Probability theory with the issue of confining a stable Lévy process to a bounded domain, which was the subject of many works such as [Sil74], [Kom95], [CK02] or [BBC03]. From a symmetric $2s$-stable Lévy process L_t, K Bogdan K. Burdzy and Z. Q. Chen define in [BBC03] three types of confined processes: the killed process L_t^{kill}, the censored process L_t^{cen} and the reflected process L_t^{ref}.
To define these processes we manipulate the associated Dirichlet form. Dirichlet forms are powerful tools, initially introduced in the field of Potential theory in the 1950s it was discovered in the 1970s that they can be directly related to certain random processes and, consequently, that they constitute a extremely useful link between probabilistic and analytic considerations. They have received a lot of attention and we refer for instance to [MMR92] for a comprehensive introduction of the subject. In the case of a Lévy flight L_t, the associated Dirichlet form is a pair $(\mathcal{E}, \mathcal{F})$ of a bilinear form $\mathcal{E}$ and a dense subspace $\mathcal{F}$ of $L^2(\mathbb{R}^d)$ such that if A is the infinitesimal generator of the semigroup associated with L_t (c.f. Section I.2.3) then for any u and v in $L^2(\mathbb{R}^d)$:

$$\mathcal{E}(u, v) = \langle -Au, v\rangle$$

where $\langle\cdot|\cdot\rangle$ is the scalar product on $L^2(\mathbb{R}^d)$, and $\mathcal{F}$ is the domain of definition of $\mathcal{E}$. In this particular case, using the results we presented as the beginning of Section I.2.3, we see immediately that the Dirichlet form associated with Lévy flights is $(\mathcal{E}, H^s(\mathbb{R}^d))$ where $\mathcal{E}$ is the scalar product on the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$

$$\mathcal{E}(u, v) = \frac{1}{2} c_{d,s} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x - y|^{d+2s}} \,\mathrm{d}x \,\mathrm{d}y.$$

Now, the construction of the killed process L_t^{kill} is the most straightforward: we add a coffin state ∂ to $\mathbb{R}^d$ and define the exit time

$$t_\Omega = \inf\{t > 0 :\ W_t \notin \Omega\}.$$

The killed process L_t^{kill} is then defined as

$$L_t^{\text{kill}} = \begin{cases} L_t & t < t_\Omega \\ \partial & t \geq t_\Omega. \end{cases}$$

It is exactly the Lévy flight killed upon leaving Ω. Its Dirichlet form $(\mathcal{E}^{\text{kill}}, \mathcal{F}_\Omega^{\text{kill}})$ is simply the Dirichlet form of the Lévy flight restricted to functions that are 0 almost everywhere outside Ω:

$$\mathcal{F}_\Omega^{\text{kill}} = \{u \in H^s(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Omega\}$$
$$\mathcal{E}^{\text{kill}}(u,v) = \frac{1}{2}c_{d,s} \iint\limits_{\Omega\times\Omega} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\Omega} u(x)v(x)\kappa_\Omega(x)\,\mathrm{d}x$$

where κ_Ω is the killing measure given by

$$\kappa_\Omega(x) = c_{d,s} \int\limits_{\mathbb{R}^d\setminus\Omega} \frac{1}{|x-y|^{d+2s}}\,\mathrm{d}y. \tag{I.79}$$

To construct the censored process L_t^{cen} we want to forbid any long flights ending outside the domain and only kill the process if it reaches the boundary through a continuous path, unlike the killed process who enters the coffin state any time it goes outside of Ω. This can be done by the Ikeda-Nagasawa-Watanabe piecing together procedure [INW66], the idea is to check, at exit time t_Ω, if the process left the domain as a result of a continuous path, in which case we kill the process with the coffin state, or if it left the domain as a result of a long flight in which case we start again with a new Lévy process initiated at $L_{t_\Omega^-}$ at time t_Ω. The resulting process is then confined to the domain Ω and reads as a juxtaposition of killed processes as defined above. K. Bogdan, K. Burdzy and Z.-Q. Chen gave an equivalent construction of this process in [BBC03] through a Dirichlet form argument and they showed that the associated form $(\mathcal{E}^{\text{cen}}, \mathcal{F}_\Omega^{\text{cen}})$ is

$$\mathcal{F}_\Omega^{\text{cen}} = H^s(\Omega) := \left\{ u \in L^2(\Omega) : \iint\limits_{\Omega\times\Omega} \frac{\big(u(x)-u(y)\big)^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty \right\}$$
$$\mathcal{E}^{\text{cen}} = \frac{1}{2}c_{d,s} \iint\limits_{\Omega\times\Omega} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

We can see, in this Dirichlet form, that forbidding any long flights ending outside the domain comes down to reducing the domain of integration to Ω, hence making "impossible" any long jumps that would result in leaving the domain.
Finally, the reflected process is basically a censored process that is not killed upon leaving the domain, instead it is reflected back inside the domain by juxtaposing censored processes. One of the fascinating results of K. Bogdan, K. Burdzy and Z.-Q. Chen is that when $0 < s < 1/2$, the reflected and the censored processes are essen-

tially identical, which means that the probability of reaching the boundary through a continuous path is 0. We will see in the chapters of this thesis that the case $s = 1/2$ is indeed critical in many situations.
The macroscopic confinement of non-local diffusion processes with PDE tools has received more and more attention in recent years, often building from the probabilistic point of view we just presented. Like at the microscopic scale, there are different ways to add a boundary condition to the fractional heat equation in order to generalise the homogeneous Dirichlet condition. The first method, which may seem the more natural from an Analysis of PDE point of view, is to impose the condition on the whole complementary of the domain:

$$\rho(t,x) = 0 \qquad (t,x) \in [0,T) \times \big(\mathbb{R}^d \setminus \Omega\big).$$

This was the subject, for instance, of [FKV13] by M. Felsinger, M. Kassmann and P. Voigt and also [ROS14] by X. Ros-Oton and J. Serra. They proved that the problem with this boundary condition is well-posed and it actually corresponds to the killed process W_t^k since it basically comes down to looking for a solution ρ of (I.78) in $H^s(\mathbb{R}^d)$ such that $\rho \equiv 0$ on the complementary of Ω.

Another way to generalise the homogeneous Dirichlet condition is to define a new operator that does not involve the values of ρ outside the domain. This operator is called the *regional fractional Laplacian* $\big(-\Delta_\Omega\big)^s$. It was introduced by Q.-Y. Guan and Z.-M. Ma in [GM06] and it is defined (for a smooth convex domain) as

$$\big(-\Delta_\Omega\big)^s \rho(x) = c_{d,s} P.V. \int_\Omega \frac{u(x)-u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y$$

which corresponds to the censored process W_t^{cen}. Since this operator confines the non-local behaviour inside the domain, it is actually compatible with local-in-position boundary conditions such as (I.70) and (I.71). There has been a growing series of work on that subject as for instance [GM05], [Gua06], [CKS09], [MY15], [War15], [War16]. Note that when considering reflective boundary conditions such as (I.71), the underlying process would be the reflected one. These papers show well-posedness of the equation and some results on the regularity of the solution up to the boundary.

There is a another remarkable approach to the problem of finding a reflective boundary condition for the fractional heat equation, which was presented by S. DiPierro, X. Ros-Oton and E. Valdinoci in [DROV17]. They propose a non-local Neumann condition in the form of an operator $\mathcal{N}_s$ defined for $x \in \mathbb{R}^d \setminus \Omega$ as

$$\mathcal{N}_x\rho(x) = c_{d,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y. \tag{I.80}$$

They prove in [DROV17] that the associated Neumann problem

$$\begin{cases} \big(-\Delta\big)^s \rho = f & x \in \Omega \\ \mathcal{N}_x \rho = 0 & x \in \mathbb{R}^d \setminus \Omega \end{cases}$$

with $f \in L^2(\Omega)$, and the fractional heat equation (I.78) with (I.80) are both well-posed and prove some properties on the solution of the heat equation like conservation of mass, decreasing energy and convergence to a steady state in long-time. The probabilistic interpretation of their operator is a variation on the censored process. Morally, their process is a juxtaposition of killed processes in such a way that when the particles jumps at a point x outside Ω, the process starts again from a point y inside the domain where y is chosen randomly with probability distribution $1/|x-y|^{d+2s}$.

I.4 List of works presented in this thesis and perspectives

In this thesis, we are interested in deriving confined non-local diffusion equations from kinetic equations with a fractional Fokker-Planck collision operator.

Chapter II: Anomalous diffusion limit with an external electric field
based on [ASC16], joint work with P. Aceves-Sànchez

In this chapter, we consider the fractional Vlasov-Fokker-Planck equation with $1/2 \leq s \leq 1$ and an external field $E(t,x)$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \nabla_v \cdot (vf) - \big(-\Delta_v\big)^s f & \text{in } [0,T) \times \mathbb{R}^d \times \mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & \text{in } \mathbb{R}^d \times \mathbb{R}^d. \end{cases}$$

We first show a result of existence of weak solutions before introducing, in the spirit of Section I.2.4, the anomalous scaling

$$t' = \varepsilon^{2s}, \qquad x' = \varepsilon x. \tag{I.81}$$

We show that if E satisfies the precise scaling property (which can be thought of as a fractional version of (I.66)):

$$E(\varepsilon^{2s} t, \varepsilon x) = \varepsilon^{2s-1} E(t', x')$$

then in the limit as ε goes to 0, f_ε will tend to $\rho(t,x)F(v)$ where ρ satisfies a fractional advection-diffusion equation

$$\begin{cases} \partial_t \rho + \nabla_x \cdot \big(E\rho\big) + \big(-\Delta\big)^s \rho = 0 & (t,x) \in [0,T) \times \mathbb{R}^d \\ \rho(0,x) = \rho_{in}(x) & x \in \mathbb{R}^d. \end{cases} \tag{I.82}$$

We conclude the chapter by focusing on the critical cases $s = 1/2$ and $s = 1$ and showing that, up to minor modifications, our method for deriving the fractional advection-diffusion equations also works in the critical cases.

Perspectives

The results presented in this chapter can be interpreted as a first step towards the anomalous diffusion limit of a Vlasov-Fokker-Planck equation with a Poisson potential.

We present some of the difficulties that arise when we consider an electric field in the kinetic equation and exhibit a sufficient (although a priori not optimal) regularity needed for that field in order to take the diffusion limit.
Note in particular, that the scaling property (I.66) does not seem compatible, at first sight, with a classical Poisson equation of the form

$$E(t,x) = \nabla_x \Phi, \qquad -\Delta\Phi = \rho.$$

As a consequence, if we want to look at a self-consistent electric field generated by the charged particles in a plasma and derive a fractional advection-diffusion equation – i.e. if we want to generalise the results of [PS00], [Gou05] or [EGM10] to the fractional case – then we should probably start by looking for a relevant generalisation of the Poisson equation that ensures the appropriate scaling property for E.

Chapter III: Anomalous diffusion limit in bounded domains

based on [Ces16]

We consider the fractional Vlasov-Fokker-Planck equation on a smooth convex bounded domain Ω

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(vf) - \big(-\Delta_v\big)^s f & \text{in } [0,T)\times\Omega\times\mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & \text{in } \Omega\times\mathbb{R}^d. \end{cases}$$

Since the non-local operator acts solely on the velocity variable, we can consider classical kinetic boundary condition on the spatial boundary $\partial\Omega$. For instance, we consider either absorption boundary condition

$$\gamma_- f(t,x,v) = 0 \qquad \text{on } [0,T)\times\Sigma_-$$

where $\Sigma_- = \{(x,v) : x\in\partial\Omega, v\cdot n(x) < 0\}$ as introduce in Section I.3.2.2, or the specular reflection boundary condition

$$\gamma_- f(t,x,v) = \gamma_+ f\big(t,x,\mathcal{R}_x(v)\big) \qquad \text{on } [0,T)\times\Sigma_-$$

where $\mathcal{R}_x(v) = v - 2v\cdot n(x)$.
We establish the anomalous diffusion limit of these problems. Introducing an anomalous scaling equivalent to (I.81), we prove a priori estimates in both cases from which we deduce the weak convergence of f_ε to $\rho(t,x)F(v)$. In the absorption case, we

identify ρ as the solution of the fractional heat equation with a Dirichlet boundary condition extended to the whole complementary of the domain

$$\begin{cases} \partial_t \rho + \big(-\Delta_x\big)^s \rho = 0 & \text{in } [0,T) \times \Omega \\ \rho(0,x) = \rho_{in}(x) & x \in \Omega \\ \rho(t,x) = 0 & \text{in } [0,T) \times \big(\mathbb{R}^d \setminus \Omega\big). \end{cases}$$

Note that in this PDE, the fractional Laplacian is to be understood as acting on the extension of ρ by 0 outside the domain.

In the specular reflection case, we show that the boundary condition affects the diffusion process of the macroscopic density inside the domain. We restrict our choice of domains to the half-space or the ball in $\mathbb{R}^d$ and, through careful analysis of the trajectories of the free transport in those domains, we construct a new non-local diffusion operator, which we call the *specular diffusion operator* and write $(-\Delta)^s_{\mathrm{SR}}$. If Ω is the half-space $\mathbb{R}^d_+ := \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > 0\}$, this operator is explicitly defined as

$$(-\Delta)^s_{\mathrm{SR}} \rho(x) = c_{d,s} P.V. \int_\Omega \big[\rho(x) - \rho(y)\big] \left(\frac{1}{|x-y|^{d+2s}} + \frac{1}{|(x'-y', x_d + y_d)|^{d+2s}} \right) \mathrm{d}y.$$

If Ω is a ball, however, the definition of $(-\Delta)^s_{\mathrm{SR}}$ involves a particular function $\eta(x,v)$ that we will construct later on and which is the focus of Appendix A. In both cases, this operator models a non-local diffusion process where the probability of jumping from x to y is not only a function of the length $|x-y|$ but also of the length of all the possible trajectories of the free transport equation that send x onto y. In the half-space, there are exactly two such trajectories, the direct jump and the reflected one, as expressed in the definition of $(-\Delta)^s_{\mathrm{SR}}$ above.

We prove some properties of this operator, define an associated generalisation of the fractional Sobolev spaces and derive an integration by parts formula. We then look at the associated evolution problem: the *specular diffusion equation*

$$\begin{cases} \partial_t \rho - (-\Delta)^s_{\mathrm{SR}} \rho = 0 & \text{in } [0,T) \times \Omega \\ \rho(0,x) = \rho_{in}(x) & x \in \Omega \end{cases}$$

and prove that this problem is well-posed in the Hilbert space associated with the operator. This results expresses the fact that the operator includes, in its definition, the interaction between the macroscopic density ρ and the boundary, which is why we don't need any extra boundary condition to have existence and uniqueness of solution.

Perspectives
The natural continuation of this work is the extension of the results to more general domains. In the case of absorption boundary condition, a similar result has been established for non-convex domains by P. Aceves-Sanchez and C. Schmeiser in [ASS17] for a linear Boltzmann operator with a heavy tailed equilibrium. They proved that the limit equation on the macroscopic density is a non-local diffusion equation where the diffusion operator resembles the regional fractional Laplacian (see Section I.3.2.3) on the largest star-shaped domain included in Ω around the point x at which it is evaluated, perturbed by a corrective term that one could compare with the killing measure (I.79) modified to adapt to the star-shaped domain, and supplemented with an homogeneous Dirichlet condition on the whole complementary of the domain. It is still an open question whether the same operator would arise as limit of the fractional Vlasov-Fokker-Planck equation although it is very likely.

In the specular reflection case, we have only been able to construct the specular diffusion operator when the domain has special symmetry properties, such as the half-space, the ball, the cube, the strip... In order to extend the construction to more general domains, or at least to prove existence of this operator in more general domain, one needs to show strong regularity results on the solution to the free transport equation in such domains which, as far as we know, remains an open question even though it has received a lot of attention.

From a stochastic point of view, we have seen in Section I.3.2.3 that in order to confine Lévy flights to a bounded domain we need to prescribe what happens when a particle leaves the domain as a result of a long flight. In the phenomenon described by the specular diffusion operator, it seems that we should see the long flight as a trajectory of the free transport equation with exponentially decreasing velocity and specularly reflected upon hitting the boundary (see Appendix A) and the Lévy motion should start again at the end-point of this trajectory. Such a Lévy process has not yet been properly constructed, as far as we know, and it would be interesting to do so in order to see if we can derive the specular diffusion operator as the infinitesimal generator of that process.

Chapter IV: Diffusion limit in spatially bounded domains

based on [CH16], joint work with H. Hutridurga

This is the only chapter where we look at classical diffusion phenomena. We consider the (classical) Vlasov-Fokker-Planck equation on a smooth convex domain Ω with specular reflection on the boundary

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (vf) + \Delta f & \text{in } [0,T) \times \Omega \times \mathbb{R}^d \\ f(0,x,v) = f_{in}(x,v) & \text{in } \Omega \times \mathbb{R}^d \\ \gamma_- f(t,x,v) = \gamma_+ f\big(t,x,\mathcal{R}_x(v)\big) & \text{in } [0,T) \times \Sigma_-. \end{cases}$$

The purpose of this chapter is to show that the method we developed for the fractional case in [CMT12] and [Ces16] also works, up to some minor changes, in the classical case. We consider the classical rescaling

$$t' = \varepsilon^2 t, \qquad x' = \varepsilon x$$

and show that the solution of the rescaled equation converges weakly to $\rho(t,x)M(v)$ where M is the gaussian equilibrium (I.31) and ρ is the unique weak solution of the heat equation with homogeneous Neumann boundary condition (I.71). This result was already known, as presented in Section I.3.2.2, with a method articulated around the current density j_ε. Here, our method is original and rests upon estimates on the regularity of trajectories described by the free transport equation in a sphere with specular reflections on the boundary.
Note that although we are only able to establish rigorously the diffusion limit in a sphere (or a half-space), we can formally derive the limit in any strongly convex open set Ω – meaning that the curvature of the boundary is bounded below by a positive constant. Extending the rigorous proof only requires a better regularity result for the free transport equation in such domains.

Chapter V: Anomalous diffusion limit with diffusive boundary
based on an ongoing project with A. Mellet and M. Puel

We consider the fractional Vlasov-Fokker-Planck equation on a smooth convex bounded domain Ω with diffusive boundary condition under anomalous rescaling:

$$\begin{cases} \varepsilon^{2s-1}\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = \dfrac{1}{\varepsilon}\Big(\nabla_v \cdot (v f_\varepsilon) - \big(-\Delta_v\big)^s f_\varepsilon\Big) & \text{in } [0,T) \times \Omega \times \mathbb{R}^d \\ f_\varepsilon(0,x,v) = f_{in}(x,v) & \text{in } \Omega \times \mathbb{R}^d \\ \gamma_- f_\varepsilon(t,x,v) = \mathcal{B}[\gamma_+ f_\varepsilon](t,x,v) & \text{on } [0,T) \times \Sigma_-. \end{cases} \tag{I.83}$$

The operator $\mathcal{B}$ is defined as

$$\mathcal{B}[\gamma_+ f](t,x,v) = c_0 F(v) \int_{\Sigma_+^x} \gamma_+ f(t,x,w) |w \cdot n(x)| \, \mathrm{d}w \tag{I.84}$$

with the normalising constant c_0 given by

$$c_0 = \Bigg(\int_{v \cdot n(x) \leq 0} F(v) |v \cdot n(x)| \, \mathrm{d}v \Bigg)^{-1}. \tag{I.85}$$

where F is the unique normalised heavy-tailed equilibrium of the fractional Fokker-Planck operator.

We prove a priori estimates on f_ε, similar to the ones established in Chapter III, that ensure convergence to the kernel of the fractional Fokker-Planck operator, i.e. to a function $\rho(t,x)F(v)$. Then, we introduce an auxiliary problem, in the spirit of Section I.2.4.2 and Chapter III, with the purpose of defining a sub-class of test functions for the weak formulation of (I.83) which will allow us to take the limit in this weak formulation. However, this auxiliary problem differs from the ones we have studied before because of the non-local nature of the boundary condition (I.76) and we are still unable, so far, to give a complete and thorough construction of its solutions.

As a consequence, we will not give a rigorous proof of the anomalous diffusion limit but instead we will identify formally the non-local diffusion equation that should be satisfied by the limit ρ which is:

$$\begin{cases} \partial_t \rho + \mathcal{L}[\rho] = 0 & \text{in } [0,T) \times \Omega \\ \rho(0,x) = \rho_{in}(x) & \text{in } \Omega \\ \mathcal{D}^{2s-1}[\rho](t,x) \cdot n(x) = 0 & \text{on } [0,T) \times \partial\Omega \end{cases} \tag{I.86}$$

where $\mathcal{L}$ is a non-local operator acting on an extension of ρ outside Ω that we define in the auxiliary problem, and $\mathcal{D}^{2s-1}$ is such that $\mathcal{L}[\rho] = -\nabla_x \cdot \mathcal{D}^{2s-1}[\rho]$.

We conclude this chapter with an analysis of this non-local operator $\mathcal{L}$, proving an integration by parts formula, defining a associated Hilbert space $\mathcal{H}^s_{\mathrm{diff}}(\Omega)$ in the spirit of the space $\mathcal{H}^s_{\mathrm{SR}}(\Omega)$, and finally proving a Poincaré-type inequality for $\mathcal{L}$.

The purpose of this chapter is two-fold. Firstly, it illustrates one of the most interesting and surprising difference between classical and anomalous diffusion limits in bounded domains which is the fact that the confined non-local diffusion equations we obtain strongly depend on whether we consider specular reflection of diffusive boundary con-

ditions, unlike the classical diffusion limits where we obtain the heat equation with homogeneous Neumann boundary condition in both cases, see Section I.2.4.2. This difference highlights the strong relation that exists between the non-local diffusion processes and the local-in-space confinement that we introduced, and the pertinence of our method to derive confined non-local diffusion equation from kinetic models.
Secondly, it allows us to introduce the new non-local operator $\mathcal{L}$ which differs from the operators presented in Section I.3.2.3. The confined stochastic processes constructed in that section do not seem to correspond to the phenomena modelled by this operator and in particular we do not know what the non-local boundary condition $\mathcal{D}^{2s-1}$ entails from a stochastic point of view. Nevertheless, the operator exhibits very promising properties. In particular we have hope to establish well-posedness of the non-local diffusion equation (I.86) in the Hilbert space $\mathcal{H}^s_{\text{diff}}(\mathbb{R}^d)$ through a Lax-Milgram argument, in the spirit of the proof for the specular diffusion case, and moreover, the Poincaré-type inequality should allow for well-posedness of the Neumann problem associated with $\mathcal{L}$ and the boundary operator $\mathcal{D}^{2s-1}$.

Perspectives

Apart from completing the proof of this anomalous diffusion limit, the natural continuation of Chapter III and this one is to consider Maxwell boundary conditions, i.e. a linear convex combination of specular reflections (I.75) and diffusive boundary condition (I.76). We see that the non-local diffusion equations we obtain in both cases are significantly different, in particular we see that the non-local operator $\mathcal{L}$ does not characterise on its own the interaction between the particle density ρ and the boundary $\partial\Omega$ and requires a boundary condition with the operator $\mathcal{D}^{2s-1}$, unlike the specular diffusion operator. As a consequence, it is not clear how we can conjugate both approaches to derive an anomalous diffusion limit for Maxwell boundary conditions, or what the resulting non-local diffusion operator will look like.

Appendix A: Free transport equation in the unit ball

based on the appendix of [Ces16] and [CH16]

In Appendix A, we have put together the appendices of [Ces16] and [CH16] concerning the free transport equation in a ball with specular reflection on the boundary. We consider the unit ball Ω in $\mathbb{R}^d$ and the trajectories $\big(x(t), v(t)\big)$ in that ball given by

$$\begin{cases} \dot{x}(t) = v(t) & x(0) = x_{in} \\ \dot{v} = -v(t) & v(0) = v_{in} \\ \text{If } x(t) \in \partial\Omega \text{ then } v(t^+) = \mathcal{R}_{x(t)}\big(v(t^-)\big) \end{cases} \tag{I.87}$$

where $\mathcal{R}$ is the specular reflection operator defined in (I.75). We define the function η – crucial in the definition of the specular diffusion operator – which associates x_{in} and v_{in} with the end-point of the trajectory described above

$$\eta(x_{in}, v_{in}) = x(t = +\infty).$$

We proved in Chapter III that if Ω is strongly convex (which is true for the unit ball) then η is well-defined, i.e. the end-point $x(t = +\infty)$ exists and is uniquely determined by x_{in} and v_{in}. Moreover, we see by construction that η is constant along the trajectories $\big(x(t), v(t)\big)$ so differentiating along the trajectories we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\eta\big(x(t), v(t)\big)\Big]\Big|_{t=0} = v \cdot \nabla_x \eta - v \cdot \nabla_v \eta = 0$$

and η satisfies the specular reflection boundary condition on $\partial\Omega$. This entails a direct link between the function η and the free-transport equation in a ball with specular reflection on the boundary. Indeed if we consider this equation with a velocity-homogeneous initial datum $\psi(x)$:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0 & (t, x, v) \in [0, T) \times \Omega \times \mathbb{R}^d \\ f(0, x, v) = \psi(x) & (x, v) \in \Omega \times \mathbb{R}^d \\ \gamma_- f(t, x, v) = \gamma_+ f(t, x, \mathcal{R}_x v) & (t, x, v) \in [0, T) \times \partial\Omega \times \{v : v \cdot n(x) < 0\} \end{cases}$$

then, it is rather straightforward to see that a solution to this problem is

$$f(t, x, v) = \psi\big(\eta(x, -tv)\big)$$

and we could also derive the solution of the free-transport equation with a non-homogeneous initial condition with respect to velocity from the η function. Hence, the regularity results of η informs us on how regularity propagates through the free transport equation in a ball, which is a long-standing open problem.

This appendix is indeed concerned with the regularity of η with respect to the velocity variable. We show that in the unit ball, η has an explicit formulation as a function of x_{in}, v_{in} and the number k of reflections on the boundary undergone by the trajectory from (x_{in}, v_{in}), which is always finite when Ω is strongly convex. Through careful manipulations of the explicit expression and elaborate differential computations, we are able to show the following regularity results:

Lemma I.4.1. *Consider the unit ball Ω. The associated function η defined above satisfies*

$$\|\nabla_v \eta(x,v)\| \in L^\infty(\Omega \times \mathbb{R}^d) \tag{I.88}$$

and for all $\psi \in \mathfrak{D}_T(\Omega)$ defined as

$$\mathfrak{D}_T(\Omega) = \Big\{\psi \in \mathcal{C}^\infty([0,T) \times \Omega) \ \textit{s.t.}\ \psi(T,\cdot) = 0 \ \textit{and}\ \forall x \in \partial\Omega : \ \nabla_x \psi(t,x) \cdot n(x) = 0\Big\}$$

we have

$$\Big\| D_v^2 \Big[\psi\big(\eta(x,v)\big)\Big]\Big\| \in L^p_{F(v)}(\Omega \times \mathbb{R}^d) \tag{I.89}$$

for any $p < 3$ where $\|\cdot\|$ is a matrix norm. Moreover,

$$\sup_{v \in \mathbb{R}^d} \Big\| D_v^2 \Big[\psi\big(\eta(x,v)\big)\Big]\Big\| \in L^{2-\delta}(\Omega) \tag{I.90}$$

for any $\delta > 0$ and finally, if we write $v = r\theta$ with $r \in \mathbb{R}^+$ and $\theta \in \mathbb{S}^{d-1}$ then

$$\sup_{r>0} \Big| \Delta \Big[\psi\big(\eta(x,\cdot)\big)\Big](r\theta)\Big| \in L^\infty\big((0,T); L^2(\Omega \times \mathbb{S}^{d-1})\big). \tag{I.91}$$

Results (I.88), (I.89) and (I.90) come from the appendix of [Ces16] and are tailor-made in order to prove convergence of the weak formulation of the fractional Vlasov-Fokker-Planck equation with a well-chosen test function. Although we do not pretend that these results are optimal, our method of proof fails to ensure the bound (I.90) for $\delta = 0$ and we have strong doubts that any better regularity could be obtained.
Result (I.91) comes from the appendix of [CH16] and is, once again, custom-made for the proof of the diffusion limit of the Vlasov-Fokker-Planck equation. Note that, although we still do not claim optimality, our proof fails to give a result uniformly in v hence the necessity to take the supremum only with respect to the norm $r = |v|$. If we took the supremum in v, the Laplacian would belong to $L^{2-\delta}(\Omega)$ for any $\delta > 0$ like the second derivative above.

Perspectives

There are two naturally continuation of this analysis. The first is to obtain a more general regularity result of η in the unit ball, such as regularity with respect to the position variable x and up to the boundary. This regularity issues have been investigate through energy estimates in [GKTT17] by Y. Guo, C. Kim, D. Tonon and A. Trescases. Together with A. Trescases, we are currently trying to combine these two approaches in order to derive optimal regularity in the particular case of the unit ball. The second natural continuation is to extend the results to more general domains. The fact is that the explicit formula for η, upon which all our analysis rests, only holds in a ball and there is a strong possibility that explicit formulae cannot be obtain in more general domains, especially ones without any particular symmetries. This is a long-standing open problem, on which we are working. Note that deriving such results would allow, almost immediately, to derive the specular diffusion equation in general domains through a very minor modification of the method presented in [Ces16].

Chapter II

Anomalous diffusion limit with an external electric field

Joint work with Pedro Aceves-Sanchez

Fractional diffusion limit for a fractional Vlasov-Fokker-Planck equation, arXiv:1606.07939, (2016).

Contents

II.1 Introduction

II.1.1 The fractional Vlasov-Fokker-Planck equation

In this chapter we investigate the long-time/small mean-free-path asymptotic behaviour in the low-field case of the solution of a Vlasov equation with a fractional

Fokker-Planck operator (VFFP) equation

$$\partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = \nabla_v\cdot(vf) - \big(-\Delta_v\big)^s f \qquad \text{in } [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d, \quad \text{(II.1a)}$$
$$f(0,x,v) = f^{in}(x,v) \qquad \text{in } \mathbb{R}^d\times\mathbb{R}^d, \quad \text{(II.1b)}$$

where $s\in[1/2,1]$. This equation describes the evolution of the density of an ensemble of particles denoted as $f(t,x,v)$ in phase space, where $t\geq 0$, $x\in\mathbb{R}^d$ and $v\in\mathbb{R}^d$ stand for, respectively, time, position and velocity. The operator $\big(-\Delta\big)^s$ denotes the fractional Laplacian and is defined by (II.5). Let us recall that, at a microscopic level, equation (II.1a)-(II.1b) is related to the Langevin equation

$$\begin{aligned} \mathrm{d}x(t) &= v(t)\,\mathrm{d}t,\\ \mathrm{d}v(t) &= -v(t)\,\mathrm{d}t + E\,\mathrm{d}t + \,\mathrm{d}L_t^{2s}, \end{aligned} \qquad \text{(II.2)}$$

where L_t^{2s} is a Lévy process with generator $-\big(-\Delta\big)^s$ and $(x(t),v(t))$ describe the position and velocity of a single particle (see [JR11] and [Ris96]). Therefore, this models describes the position and velocity of a particle that is affected by three mechanisms: a dragging force, an acceleration and a pure jump process.
In the particular case when $s=1$ the fractional operator $\big(-\Delta\big)^s$ takes the form of a Laplace operator $-\Delta$ and (II.1a)-(II.1b) reduces to the usual Vlasov-Fokker-Planck equation. In this case the Fokker-Planck operator is known to have an equilibrium distribution function given by a Maxwellian $M(v) = C\exp(-|v|^2)$ where $C>0$ is a normalization constant. The Vlasov-Fokker-Planck equation has been used in the modeling of many physical phenomena, in particular, for the description of the evolution of plasmas [Ris96]. However, there are some settings in which particles may have long jumps and an $2s$-stable distribution process is more suitable to describe the phenomenon, see for instance [SLD+01]. The classical Vlasov-Fokker-Planck equation for a given external field is related to the Vlasov-Poisson-Fokker-Planck system (VPFP) in the case in which the electric field is self-consistent. Questions such as existence of solutions, hydrodynamic limits and long time behaviour for the VPFP system has been extensively studied by many authors, see for instance [BD95], [Pfa92], and [GNPS05]. In particular, in [EGM10] the low field limit is studied for the VPFP system and a Drift-Diffusion-Poisson system is obtained in a rigorous manner.
Let us note that, although it is classical in the framework of kinetic theory to consider a self-consistence electric fields that expresses how particles repulse one another, one can also, in the VPFP system, consider the case in which particles are attracted by each other and this model is used in the description of galactic dynamics.

In the rest of the chapter we shall need the following notation: The fractional (or Lévy) Fokker-Planck operator denoted by $\mathcal{L}^s$ and defined as

$$\mathcal{L}^s f = \nabla_v \cdot \big(vf\big) - \big(-\Delta_v\big)^s f. \tag{II.3}$$

In order to investigate the asymptotic behaviour of the system, we introduce the Knudsen number ε which represent the ratio between the mean-free-path and the observation length scale. In the case when $E = 0$ it was observed in [CMT12] that the time rescaling $t' \to \varepsilon^{2s-1}t$ and introducing a factor $1/\varepsilon$ in front of $\mathcal{L}^s$ is the appropriate scaling at which diffusion will be observed in the limit as ε goes to zero. Moreover, we introduce the factor $1/\varepsilon^{2-2s}$ in front of the force field term E corresponding to a low-field limit scaling since we shall consider the case $1/2 \le s \le 1$ and thus the scaling of the collision operator $1/\varepsilon$ is much greater than the scaling of the electric field $1/\varepsilon^{2-2s}$. We shall study in this paper the asymptotic behaviour as ε tends to zero of the solutions of following rescaled VLFP equation

$$\varepsilon^{2s-1}\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \varepsilon^{2s-2}E(t,x)\cdot\nabla_v f^\varepsilon = \frac{1}{\varepsilon}\Big(\nabla_v\cdot(vf) - \big(-\Delta_v\big)^s f\Big). \tag{II.4}$$

II.1.2 Preliminaries on the Fractional Fokker-Planck operator

In this chapter we denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of f and define it as

$$\widehat{f}(k) = \int_{\mathbb{R}^d} \mathrm{e}^{-ik\cdot x} f(x)\,\mathrm{d}x.$$

There are several equivalent definitions of the fractional Laplacian in the whole domain (see [Kwa15] or [DPV12]). It can be defined via a Fourier multiplier as

$$\mathcal{F}\Big(\big(-\Delta\big)^s(f)\Big)(k) = |k|^{2s}\mathcal{F}(f)(k).$$

On the other hand, assuming that f is a rapidly decaying function we can define the fractional Laplacian in terms of a hypersingular integral as

$$\big(-\Delta_v\big)^s(f)(v) = c_{d,s}\,\mathrm{P.V.}\int_{\mathbb{R}^d}\frac{f(v)-f(w)}{|v-w|^{d+2s}}\,\mathrm{d}w \tag{II.5}$$

where P.V. denotes the Cauchy principal value and the constant $c_{d,s}$ is given by

$$c_{d,s} = \frac{2^{2s}\Gamma\left(\frac{d+2s}{2}\right)}{2\pi^{d/2}|\Gamma\left(-s\right)|}, \tag{II.6}$$

and $\Gamma(\cdot)$ denotes the Gamma function. In [DPV12] it is proven that for any $d > 1$, $c_{d,s} \to 0$ as $s \to 1$. Thus (II.5) does not make sense if we take $s = 1$. However, we have the following result.

Proposition II.1.1. *Let $d > 1$. Then for any $f \in C_0^\infty(\mathbb{R}^d)$ we have*

$$\lim_{s\to 1} \big(-\Delta\big)^s f = -\Delta f.$$

For an account of the properties of the fractional Laplacian consult [DPV12], [VÍ14], [Ste70] or [Lan72]. Let us note that due to its dependence on the whole domain, the fractional Laplacian is a nonlocal operator and it has the scaling property $\big(-\Delta_v\big)^s(f_\lambda)(v) = \lambda^{2s}\big(-\Delta_v\big)^s f(\lambda v)$, for any $\lambda > 0$ where $f_\lambda(v) = f(\lambda v)$. Since it will be useful later on in our analysis, we also mention that since the fractional Laplacian is an integro-differential operator it satisfies:

$$\int \big(-\Delta\big)^s f \,\mathrm{d}v = 0.$$

In [BK03] it is proved that the Lévy-Fokker-Planck operator $\mathcal{L}^s$ defined by (II.3) has a unique normalized equilibrium distribution that we shall denote F (which depends on s). Therefore, the Fourier transformation of F denoted as $\widehat{F}$ and defined as

$$\widehat{F}(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-i\xi\cdot v} F(v)\,\mathrm{d}v,$$

satisfies

$$\xi \cdot \nabla_\xi \widehat{F} + |\xi|^{2s}\widehat{F} = 0.$$

Thus yielding

$$\widehat{F}(\xi) = \mathrm{e}^{-|\xi|^{2s}/2s}. \tag{II.7}$$

In the jargon of stochastic analysis, random variables having a characteristic function of the form (II.7) are called symmetric $2s$-stable random variables, consult [App09]. Using the notation of [BJ07] let us note that setting $t = 1/2s$, $x = v$, and $y = 0$, we obtain the identity $F(v) = p(1/2s, v, 0)$. Thus Lemma 3 of [BJ07] states that there exists $C_1 = C_1(d,s) > 0$ such that

$$C_1^{-1}\left(\frac{1}{2s|v|^{d+2s}} \wedge \frac{1}{(2s)^{d/2s}}\right) \le F(v) \le C_1\left(\frac{1}{2s|v|^{d+2s}} \wedge \frac{1}{(2s)^{d/2s}}\right), \tag{II.8}$$

for all $v \in \mathbb{R}^d$, where $a \wedge b$ denotes the minimum between a and b. On the other hand, Lemma 5 of [BJ07] states the existence of a positive constant $C_2 = C_2(d,s)$ such that

$$\frac{|v|}{C_2}\left(\frac{1}{2s|v|^{d+2+2s}} \wedge (2s)^{(d+2)/2}\right) \leq \left|\nabla_v F(v)\right| \leq C_2|v|\left(\frac{1}{2s|v|^{d+2+2s}} \wedge (2s)^{(d+2)/2}\right). \tag{II.9}$$

II.1.3 Main results

As usually in the framework of fractional Vlasov-Fokker-Planck equations, we use the following definition of weak solutions:

Definition II.1.1. *Consider f^{in} in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $E \in \left(W^{1,\infty}([0,T) \times \mathbb{R}^d)\right)^d$. We say that f is a weak solution of* (II.1a)-(II.1b) *if, for any $\phi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$*

$$\begin{aligned}\iiint\limits_{Q_T} f\Big(\partial_t\phi + v\cdot\nabla_x\phi + \big(E(t,x)-v\big)\cdot\nabla_v\phi - \big(-\Delta\big)^s\phi\Big)\,\mathrm{d}txv \\ + \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\phi(0,x,v)\,\mathrm{d}xv = 0\end{aligned} \tag{II.10}$$

where $Q_T := [0,T) \times \mathbb{R}^d \times \mathbb{R}^d$. Section 2 of this chapter is devoted to a well-posedness result for the fractional Vlasov-Fokker-Planck with an external electric field E in the following sense.

Theorem II.1.2. *For f^{in} in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and $E \in \left(W^{1,\infty}([0,T) \times \mathbb{R}^d)\right)^d$ there exists a unique weak solution f of* (II.1a)-(II.1b) *in the sense of Definition II.1.1 and it satisfies*

$$f(t,x,v) \geq 0 \ \text{ on } Q_T, \tag{II.11a}$$

$$f \in \mathcal{X} := \left\{ f \in L^2(Q_T) : \frac{|f(t,x,v)-f(t,x,w)|}{|v-w|^{\frac{d+2s}{2}}} \in L^2(Q_T \times \mathbb{R}^d) \right\}. \tag{II.11b}$$

Remark II.1.3. *The assumption $E \in \left(W^{1,\infty}([0,T) \times \mathbb{R}^d)\right)^d$ in Theorem II.1.2 is not optimal in the sense that we could replace it by $E \in \left(L^\infty([0,T) \times \mathbb{R}^d)\right)^d$ or maybe it could be replaced by even weaker assumptions on E, however, finding the optimal regularity of E is out of the scope of this chapter.*

The proof of this existence result relies on using the Lax-Milgram theorem for a well chosen associated problem, in the spirit of the proof in [Deg86] and in [Car98] for the existence of weak solutions of the Vlasov-Fokker-Planck equation. The proof of

positivity (II.11a) is given in details as it involves the non-local nature of the fractional operator and, therefore, differs from the classical proof.
In Section 3, we consider the electric field as a perturbation of the fractional Fokker-Planck operator and as such we introduce $\mathcal{T}_\varepsilon$:

$$\mathcal{T}_\varepsilon(f) := \nabla_v \cdot \Big[\big(v - \varepsilon^{2s-1}E(t,x)\big)f\Big] - \big(-\Delta_v\big)^s f.$$

We prove existence and uniqueness of a normalized equilibrium F_ε for this perturbed operator in Proposition II.3.1. Then, following the strategy introduced in [ASS17], we investigate the decay properties of this equilibrium and its convergence to the equilibrium of the unperturbed operator, F, as ε goes to 0, in Proposition II.3.2. Finally, we prove that $\mathcal{T}_\varepsilon$ is dissipative with regards to the quadratic entropy, Proposition II.3.3, which allows us to establish uniform boundedness results for f_ε, the solution of the rescaled equation (II.4)-(II.1b), as well as its macroscopic density $\rho_\varepsilon = \int f_\varepsilon \,\mathrm{d}v$ and its distance to the kernel of $\mathcal{T}_\varepsilon$ we which write r_ε defined by the expansion $f_\varepsilon = \rho_\varepsilon F_\varepsilon + \varepsilon^s r_\varepsilon$.
In the last section, we turn to the proof of our main result which is the anomalous advection-diffusion limit of our kinetic model. We follow the method introduced in [CMT12] which consist in choosing a test function $\phi_\varepsilon(t,x,v)$ which is solution, for some $\psi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$ of the auxiliary problem:

$$\begin{aligned}
&\varepsilon v\cdot\nabla_x\phi_\varepsilon - v\cdot\nabla_v\phi_\varepsilon = 0 &&\text{in } [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\\
&\phi_\varepsilon(t,x,0) = \psi(t,x) &&\text{in } [0,\infty)\times\mathbb{R}^d,
\end{aligned}$$

and show that the weak formulation of our problem, (II.14), with such test functions converges to the weak formulation of the advection fractional diffusion equation. We first prove this convergence in the non-critical case, i.e. when $1/2 < s < 1$ and then we turn to the critical cases $s = 1/2$ and $s = 1$. The outline of the proof remains the same in both critical cases but a few differences appear. For $s = 1$, although the nature of the collision operator changes noticeably since it becomes local, the only difference in the proof is a technical one in the study of the dissipative property of the perturbed operator whereas, in the case $s = 1/2$, we show that the equilibrium of the perturbed operator is independent of ε and as such it stays perturbed by the electric field $E(t,x)$ even in the macroscopic limit. In all cases, our main result reads:

Theorem II.1.4. *Let s be in $(1/2,1]$ and f_ε be the weak solution of* (II.4)-(II.1b) *in the sense of Definition II.1.1 on $[0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ for some $T > 0$ and with $f^{in} \in L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d)\cap L^1_+(\mathbb{R}^d\times\mathbb{R}^d)$. Then, f_ε converges weak-$*$ to $\rho(t,x)\,F(v)$*

in $L^\infty(0,T;L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))$*, where* ρ *is the solution in the distributional sense of*

$$\begin{aligned} &\partial_t\rho+\nabla\cdot(E\rho)+\big(-\Delta\big)^s\rho=0 \quad &&\textit{in } [0,T)\times\mathbb{R}^d,\\ &\rho(0,x)=\rho^{in}(x) &&\textit{in } \mathbb{R}^d, \end{aligned} \tag{II.12}$$

where $\rho^{in}=\int f^{in}\,\mathrm{d}v$*. In the case* $s=1/2$ *the same anomalous advection-diffusion limit holds but instead of* $F(v)$ *the equilibrium distribution of velocity becomes*

$$F_E(t,x,v)=F\big(v-E(t,x)\big) \tag{II.13}$$

The advection fractional-diffusion equation (II.12) describes the evolution of the macroscopic density ρ under the effect of a drift, consequence of the kinetic electric field, and a fractional diffusion phenomenon. The regularity of the solutions of this type of equations has been studied for instance in [Sil11], [Sil12], and [DI06]. We refer the interested reader to those articles and references within for more details on this macroscopic model.

II.2 Existence of solution

We recall that, throughout this paper, for any $T>0$ we write $Q_T=[0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ and $\mathcal{C}^\infty_c(Q_T)$ the set of smooth function compactly supported in Q_T. This section is devoted to the proof of the following result of existence and regularity of weak solutions:

Theorem II.2.1. *Consider* f^{in} *in* $L^2(\mathbb{R}^d\times\mathbb{R}^d)$*. There exists a unique weak solution* f *of* (II.1a)-(II.1b) *on* Q_T *in the sense that for any* $\phi\in\mathcal{C}^\infty_c(Q_T)$*:*

$$\begin{aligned} &\iiint_{Q_T} f\Big(\partial_t\phi+v\cdot\nabla_x\phi+\big(E(t,x)-v\big)\cdot\nabla_v\phi-\big(-\Delta\big)^s\phi\Big)\,\mathrm{d}txv\\ &\quad+\iint_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\phi(0,x,v)\,\mathrm{d}xv=0 \end{aligned} \tag{II.14}$$

and this solution satisfies:

$$\begin{aligned} &f(t,x,v)\geq 0 \ \textit{ on } Q_T,\\ &f\in\mathcal{X}:=\left\{f\in L^2(Q_T):\frac{|f(t,x,v)-f(t,x,w)|}{|v-w|^{\frac{d+2s}{2}}}\in L^2(Q_T\times\mathbb{R}^d)\right\}. \end{aligned} \tag{II.15}$$

Remark II.2.2. *Note that this definition of $\mathcal{X}$ is equivalent to saying that it is the set of functions which are in $L^2([0,T)\times\mathbb{R}^d)$ with respect to time and position and in $H^s(\mathbb{R}^d)$ with respect to velocity.*

Proof. We follow the method in [Deg86] and in [Car98] for the proof of existence and uniqueness of solutions to the linear Vlasov-Fokker-Planck equation. The first part of the proof consists in solving our linear problem in a variational setting, applying the well-known Lax-Milgram theorem of functional analysis. We consider the Hilbert space $\mathcal{X}$ provided with the norm

$$||f||_{\mathcal{X}} = \left(||f||^2_{L^2(Q_T)} + 2c^{-1}_{d,s}||(-\Delta)^{\frac{s}{2}} f||^2_{L^2(Q_T)} \right)^{\frac{1}{2}} \tag{II.16}$$

where $c_{d,s}$ is defined in (II.6). We refer the reader to [DPV12] for properties of this functional space. Let us denote $\mathcal{T}$ the transport operator, given by

$$\mathcal{T}f = \partial_t f + v\cdot\nabla_x f - \big(v - E(t,x)\big)\cdot\nabla_v f.$$

We define the Hilbert space $\mathcal{Y}$ as:

$$\mathcal{Y} = \Big\{ f\in\mathcal{X} : \mathcal{T}f \in \mathcal{X}' \Big\} \tag{II.17}$$

where $\mathcal{X}'$ is the dual of $\mathcal{X}$. $(\cdot,\cdot)_{\mathcal{X},\mathcal{X}'}$ stands for the dual relation between $\mathcal{X}$ and its dual. $\mathcal{Y}$ is provided with the norm:

$$||f||^2_{\mathcal{Y}} = ||f||^2_{\mathcal{X}} + ||\mathcal{T}f||^2_{\mathcal{X}'}. \tag{II.18}$$

In order to apply the Lax-Milgram theorem we consider the associated problem

$$\begin{aligned} &\partial_t\bar{f} + e^{-t}v\cdot\nabla_x\bar{f} + e^t E(t,x)\cdot\nabla_v\bar{f} + e^{2st}\big(-\Delta\big)^s\bar{f} + \lambda\bar{f} = 0 && (t,x,v)\in Q_T \\ &\bar{f}(0,x,v) = \bar{f}^{in}(x,v) && (x,v)\in\mathbb{R}^d\times\mathbb{R}^d \end{aligned} \tag{II.19}$$

which comes formally by deriving (II.1a) for $\bar{f} = e^{-(\lambda+d)t}f\big(t,x,e^{-t}v\big)$ and $\bar{f}^{in}(x,v) = f^{in}(x,e^{-t}v)$ for some $\lambda\geq 0$. A weak solution of (II.19) is a function $\bar{f}\in\mathcal{X}$ such that for any ϕ in $\mathcal{C}_c^\infty(Q_T)$:

$$\iiint\limits_{Q_T} \Big(- \bar{f}\partial_t\phi - e^{-t}\bar{f}v \cdot \nabla_x\phi - e^t\bar{f}E(t,x) \cdot \nabla_v\phi + e^{2st}\bar{f}\big(- \Delta\big)^s\phi + \lambda\bar{f}\phi\Big) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \\ - \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \bar{f}^{in}\phi(0,x,v) \, \mathrm{d}x \, \mathrm{d}v = 0. \tag{II.20}$$

We first prove existence of a solution in $\mathcal{X}$ of equation (II.19) and we will prove afterwards how this implies existence of a solution of the fractional Vlasov-Fokker-Planck equation with the electric field E.

We know that $\mathcal{C}_c^\infty(Q_T)$ is a subspace of $\mathcal{X}$ with a continuous injection (see, e.g. [DPV12]) and we define the prehilbertian norm:

$$|\phi|^2_{\mathcal{C}_c^\infty(Q_T)} = ||\phi||^2_{\mathcal{X}} + \frac{1}{2}||\phi(0,\cdot,\cdot)||^2_{L^2(\Omega\times\mathbb{R}^d)}.$$

Now, we can introduce the bilinear form $a : \mathcal{X} \times \mathcal{C}_c^\infty(Q_T) \to \mathbb{R}$ as:

$$a(\bar{f},\phi) = \iiint\limits_{Q_T} \Big(- \bar{f}\partial_t\phi - e^{-t}\bar{f}v \cdot \nabla_x\phi - e^t\bar{f}E(t,x) \cdot \nabla_v\phi + e^{2st}\bar{f}\big(- \Delta\big)^s\phi + \lambda\bar{f}\phi\Big) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v$$

and the continuous bounded linear operator L on $\mathcal{C}_c^\infty(Q_T)$ given by:

$$L(\phi) = - \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \bar{f}^{in}(x,v)\phi(0,x,v) \, \mathrm{d}x \, \mathrm{d}v.$$

To find a solution $\bar{f}$ in $\mathcal{X}$ of equation (II.20) is equivalent to finding a solution $\bar{f}$ in $\mathcal{X}$ of $a(\bar{f},\phi) = L(\phi)$ for any $\phi \in \mathcal{C}_c^\infty(Q_T)$. Since $\bar{f}$ belongs to $\mathcal{X}$ it is easy to check that $a(\cdot,\phi)$ is continuous. To verify the coercivity of a we write:

$$- \iiint\limits_{Q_T} \Big(\phi\partial_t\phi + e^{-t}\phi v \cdot \nabla_x\phi - e^t\phi E(t,x) \cdot \nabla_v\phi\Big) \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v = \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} |\phi(0,x,v)|^2 \, \mathrm{d}x \, \mathrm{d}v$$

and also:

$$\iiint\limits_{Q_T} e^{2st}\phi\big(- \Delta\big)^s\phi \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v = \iiint\limits_{Q_T} e^{2st}|(-\Delta)^{\frac{s}{2}}\phi|^2 \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v.$$

Hence, we see that

$$a(\phi,\phi)=\iiint\limits_{Q_T}\Big(\lambda\phi^2+e^{2st}|(-\Delta)^{\frac{s}{2}}\phi|^2\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v+\frac{1}{2}\iint\limits_{\Omega\times\mathbb{R}^d}|\phi(0,x,v)|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$

which can be bounded from below as $a(\phi,\phi)\geq\min(1,\lambda)|\phi|^2_{\mathcal{C}^\infty_c(Q_T)}$. Thus, the Lax-Milgram theorem implies the existence of $\bar{f}$ in $\mathcal{X}$ satisfying (II.20). Now, we want to show that this yields existence of a solution of (II.14). To that end, we first consider $\tilde{\phi}$ in $\mathcal{C}^\infty_c(Q_T)$ such that $\phi(t,x,v)=e^{\lambda t}\tilde{\phi}(t,x,e^{-t}v)$. Equation (II.20) becomes (writing $\tilde{\phi}(e^{-t}v)$ instead of $\tilde{\phi}(t,x,e^{-t}v)$)

$$\begin{aligned}\iiint\limits_{Q_T}e^{\lambda t}\Big(&-\bar{f}\partial_t\tilde{\phi}(e^{-t}v)-\bar{f}e^{-t}v\cdot\nabla_x\tilde{\phi}(e^{-t}v)+\bar{f}e^{-t}v\cdot\nabla_v\tilde{\phi}(e^{-t}v)\\&-\bar{f}E(t,x)\cdot\nabla_v\tilde{\phi}(e^{-t}v)+\bar{f}\big(-\Delta\big)^s\tilde{\phi}(e^{-t}v)\Big)\,\mathrm{d}txv-\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}f_{in}\tilde{\phi}(0,x,v)\,\mathrm{d}xv=0.\end{aligned}$$

Hence, if we define $f(t,x,v)=e^{(\lambda+d)t}\bar{f}(t,x,e^tv)$ and change the variable $v\to e^{-t}v$, we recover equation (II.14). It is straightforward to check that f is in $\mathcal{X}$ and it satisfies (II.14) for any $\tilde{\phi}$ in $\mathcal{C}^\infty_c(Q_T)$. Moreover, since $f\mapsto df-\big(-\Delta\big)^sf$ is a linear bounded operator from $\mathcal{X}$ to $\mathcal{X}'$, the transport term $\mathcal{T}f$ is in $\mathcal{X}'$, hence $f\in\mathcal{Y}$ and (II.14) is verified in $\mathcal{X}'$.

Since the VLFP equation is linear, to show uniqueness it is enough to show that the unique solution with zero initial data is the null function $f\equiv 0$. Let f be a solution of this problem on $\mathcal{Y}$. As before, we define $\bar{f}=e^{-(\lambda+d)t}f(t,x,e^{-t}v)$, which satisfies equation (II.19) with $\bar{f}_{in}$ null. Since $f\in\mathcal{Y}$, we know that $\bar{f}$ belongs to $\mathcal{X}$ and, moreover, that if we define $\widetilde{\mathcal{T}}$ as

$$\widetilde{\mathcal{T}}\bar{f}=\partial_t\bar{f}+e^{-t}v\cdot\nabla_x\bar{f}+e^tE(t,x)\cdot\nabla_v\bar{f}\tag{II.21}$$

then $\widetilde{\mathcal{T}}\bar{f}$ belongs to $\mathcal{X}'$. Through integration by parts we have

$$2\big(\widetilde{\mathcal{T}}\bar{f},\bar{f}\big)_{\mathcal{X}',\mathcal{X}}=\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\big(\bar{f}\big)^2(T,x,v)\,\mathrm{d}x\,\mathrm{d}v\geq 0.$$

On the other hand, since $\bar f$ satisfies (II.19), $\widetilde{\mathcal{T}}\bar f = -\lambda \bar f - \big(-\Delta\big)^s \bar f$ in the sense of distributions which yields

$$\big(\widetilde{\mathcal{T}}\bar f, \bar f\big)_{\mathcal{X}',\mathcal{X}} = -\iiint\limits_{Q_T} \Big(\lambda \bar f^2 + e^{2st}\big|(-\Delta)^{\frac{s}{2}}\bar f\big|^2\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \le 0. \tag{II.22}$$

Hence both expression are null, in particular this means that the integral $\lambda \bar f^2$ is null, hence $f = \bar f \equiv 0$ a.e. on Q_T: the solution is unique. In order to prove the positivity of the solution consider once again the associated problem (II.19) and its solution $\bar f$ for some $\bar f^{in} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ with $\bar f^{in} \ge 0$. Next, we define $\bar f_+$ and $\bar f_-$ the positive and negative parts of $\bar f$ given by:

$$\bar f_+(t,x,v) = \max(f(t,x,v),0); \qquad \bar f_-(t,x,v) = \max(-f(t,x,v),0)$$

so that $\bar f = \bar f_+ - \bar f_-$ and we denote by A_+ and A_- the respective supports of $\bar f_+$ and $\bar f_-$. Using $\widetilde{\mathcal{T}}$ defined in (II.21) we have through integration by parts

$$\begin{aligned}
\big(\widetilde{\mathcal{T}}\bar f, \bar f_-\big) = &\iiint\limits_{Q_T} \Big(\bar f_- \partial_t\big(\bar f_+ - \bar f_-\big) + e^{-t}\bar f_- v\cdot\nabla_x\big(\bar f_+ - \bar f_-\big) \\
&\qquad + e^t \bar f_- E(t,x)\cdot\nabla_v\big(\bar f_+ - \bar f_-\big)\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\
= &-\frac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \Big(\bar f_-^2(T,x,v) - \bar f_-^2(0,x,v)\Big)\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad + \iiint\limits_{Q_T}\Big(\bar f_-\partial_t \bar f_+ + e^{-t}\bar f_- v\cdot\nabla_x \bar f_+ + e^t\bar f_- E(t,x)\cdot\nabla_v\bar f_+\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

By definition of $\bar f_+$ and $\bar f_-$ we know that $A_+ \cap A_- = \emptyset$, hence wherever $\bar f_-$ is not zero, both $\partial_t \bar f_+$, $\nabla_x \bar f_+$ and $\nabla_v \bar f_+$ are naught, and vice-versa. Moreover, we assume $\bar f^{in} \ge 0$ which means $\bar f_-(0,x,v) = 0$ so that

$$\big(\widetilde{\mathcal{T}}\bar f, \bar f_-\big) = -\frac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \bar f_-^2(T,x,v)\,\mathrm{d}x\,\mathrm{d}v \le 0.$$

Since $\bar{f}$ is solution of (II.19) we know that $\widetilde{\mathcal{T}}\bar{f} = -\lambda\bar{f} - \big(-\Delta\big)^s\bar{f}$ in the sense of distributions which yields

$$\big(\widetilde{\mathcal{T}}\bar{f},\bar{f}_-\big) = \iiint\limits_{Q_T}\Big(-\lambda\bar{f}_-\big(\bar{f}_+ - \bar{f}_-\big) - \bar{f}_-\big(-\Delta\big)^s\big(\bar{f}_+ - \bar{f}_-\big)\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$$

where

$$\begin{aligned}\int\limits_{\mathbb{R}^d}\bar{f}_-\big(-\Delta\big)^s(\bar{f}_+)\,\mathrm{d}v &= \int\limits_{\mathbb{R}^d}\bar{f}_-(v)\,c_{d,s}\ \text{P.V.}\int\limits_{\mathbb{R}^d}\frac{\bar{f}_+(v)-\bar{f}_+(w)}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}v\\ &= \int\limits_{A_-}\bar{f}_-(v)\,c_{d,s}\ \text{P.V.}\int\limits_{A_+}\frac{\bar{f}_+(v)-\bar{f}_+(w)}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}v\\ &= -c_{d,s}\int\limits_{A_-}\ \text{P.V.}\int\limits_{A_+}\frac{\bar{f}_-(v)\bar{f}_+(w)}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}v \le 0.\end{aligned}$$

Note that this integral is well defined because $\bar{f}\in\mathcal{X}$. Hence, we have:

$$\big(\widetilde{\mathcal{T}}\bar{f},\bar{f}_-\big) = \iiint\limits_{Q_T}\Big(\lambda\bar{f}_-^2 - \bar{f}_-\big(-\Delta\big)^s\bar{f}_+ + \big|(-\Delta)^{s/2}\bar{f}_-\big|^2\Big)\,\mathrm{d}txv \ge 0.$$

This proves that $\big(\widetilde{\mathcal{T}}\bar{f},\bar{f}_-\big) = 0$ which, in particular, means $\lambda\bar{f}_-^2 = 0$ and concludes the proof of positivity, and consequently the proof of Theorem II.2.1. □

II.3 A priori estimates

Let us consider the operator $\mathcal{T}_\varepsilon$: a perturbation of the fractional Fokker-Planck operator with an electric field $E(t,x)\in\big(W^{1,\infty}([0,T)\times\mathbb{R}^d)\big)^d$ defined as

$$\mathcal{T}_\varepsilon(f_\varepsilon) = \nabla_v\cdot\Big[\big(v-\varepsilon^{2s-1}E(t,x)\big)f_\varepsilon\Big] - \big(-\Delta_v\big)^s f_\varepsilon. \tag{II.23}$$

We will prove the following:

Proposition II.3.1. *For any $\varepsilon > 0$ fixed, there exists a unique positive equilibrium distribution F_ε solution of:*

$$\mathcal{T}_\varepsilon(F_\varepsilon) = \nabla_v\cdot\Big[\big(v-\varepsilon^{2s-1}E(t,x)\big)F_\varepsilon\Big] - \big(-\Delta_v\big)^s F_\varepsilon = 0,\qquad \int_{\mathbb{R}^d}F_\varepsilon\,\mathrm{d}v = 1. \tag{II.24}$$

Proof. The Fourier transform in velocity of the equilibrium equation (II.24) reads

$$\xi \cdot \nabla_\xi \widehat{F_\varepsilon} = -\Big(i\xi \cdot \varepsilon^{2s-1} E(t,x) + |\xi|^{2s}\Big)\widehat{F_\varepsilon},$$

for which we can compute the explicit solution:

$$\widehat{F_\varepsilon}(t,x,\xi) = \kappa e^{-i\varepsilon^{2s-1}\xi\cdot E(t,x) - |\xi|^{2s}/2s}, \tag{II.25}$$

where κ is a positive constant which ensures the normalisation of the equilibrium. Now, although the inverse Fourier transform $\mathcal{F}^{-1}\big(\widehat{F_\varepsilon}\big)(t,x,v)$ is not explicit let us note that F_ε can be expressed as a translation of the equilibrium distribution F of the fractional Fokker-Planck operator:

$$F_\varepsilon(t,x,v) = F\big(v - \varepsilon^{2s-1}E(t,x)\big). \tag{II.26}$$

Hence, the positivity and normalization of F_ε follows from the properties of F. □

Proposition II.3.2. *Let F_ε be the unique normalized equilibrium distribution of* (II.23)*. Then there exist positive constants μ, c_1, c_2 and c_3 such that:*

(i) $c_1 F \le F_\varepsilon \le c_2 F$,

(ii) $\displaystyle \left\|\frac{\partial_t F_\varepsilon}{F_\varepsilon}\right\|_{L^\infty(\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)}, \left\|\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon}\right\|_{L^\infty(\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t)} \le \varepsilon^{2s-1}\mu,$

(iii) $|F_\varepsilon - F| \le \varepsilon^{2s-1} c_3 F$.

for $\varepsilon > 0$ small enough.

Proof. We shall start by proving part (i). Let us assume that L is an arbitrary vector in $\mathbb{R}^d$ such that $|L| \le 1$, then is easy to see that there exists $R_1 > 0$ big enough such that for all $|v| > R_1$

$$\frac{1}{2^{\frac{1}{d+2s}}} \le \left|1 - \frac{|L|}{|v|}\right| \le \left|\frac{v}{|v|} - \frac{L}{|v|}\right|.$$

Hence, it follows that

$$\frac{1}{|v-L|^{d+2s}} \le \frac{2}{|v|^{d+2s}},$$

for all $|v| > R_1$. Thus, using (II.8) we obtain that there exists $\widetilde{C} > 0$ and $R > 0$ big enough such that

$$F(v-L) \leq \widetilde{C} F(v),$$

for all $|v| > R$ and all $L \in \mathbb{R}^d$ with $L \leq 1$. Now, let us consider $C_2 > 0$ such that

$$C_2 \left(\min_{v \in B(0,R)} F(v) \right) \geq \|F\|_\infty,$$

where $B(0,R) \subset \mathbb{R}^d$, is the ball of radius R centered at the origin. Let us note that the minimum exists since F is continuous. Thus choosing $\mu_2 = \widetilde{C} \vee C_2$, where $a \vee b$ denotes the maximum between a and b, we obtain

$$F(v-L) \leq \mu_2 F(v).$$

Next, writing $w = v + L$ where $L \in \mathbb{R}^d$ with $|L| \leq 1$ we obtain

$$F(w) \leq \mu_1 F(w-L),$$

Thus, taking $\mu_1 = 1/\mu_2$ we obtain

$$\mu_1 F(v) \leq F(v-L),$$

for all $v \in \mathbb{R}^d$ and $|L| \leq 1$.
On the other hand, for part (ii), let us start by noting that thanks to (II.26), F_ε satisfies the following identities:

$$\frac{\partial_t F_\varepsilon}{F_\varepsilon} = -\varepsilon^{2s-1} \partial_t E(t,x) \cdot \frac{\nabla_v F\big(v - \varepsilon^{2s-1} E(t,x)\big)}{F\big(v - \varepsilon^{2s-1} E(t,x)\big)},$$

and

$$\frac{v \cdot \nabla_x F_\varepsilon}{F_\varepsilon} = -\varepsilon^{2s-1} \nabla_x E(t,x) \frac{v \cdot \nabla_v F\big(v - \varepsilon^{2s-1} E(t,x)\big)}{F\big(v - \varepsilon^{2s-1} E(t,x)\big)}.$$

Hence, thanks to the assumption $E \in \big(W^{1,\infty}([0,T) \times \mathbb{R}^d)\big)^d$ we only need to prove that there exists a $C > 0$ such that

$$|v \cdot \nabla_v F(v-L)| \leq C F(v-L), \tag{II.27}$$

for all $v \in \mathbb{R}^d$, and all $L \in \mathbb{R}^d$ with $|L| \leq 1$. This follows via a similar line of reasoning as in the proof of part (i) around the control (II.9).
Finally we prove part (iii). Since F is smooth by the mean value theorem we obtain

$$
\begin{aligned}
|F_\varepsilon(v) - F(v)| &= |F(v - \varepsilon^{2s-1}E) - F(v)| \\
&= \varepsilon^{2s-1}|E||\nabla_v F(v - \vartheta\, \varepsilon^{2s-1}E)|,
\end{aligned}
$$

where $\vartheta \in (0,1)$. Thus, the result follows thanks to (II.27) and since $E \in \big(W^{1,\infty}([0,T)\times \mathbb{R}^d)\big)^d$.

□

The key ingredient in order to obtain the a priori estimates needed to pass to the limit in (II.4) is the positivity of the dissipation which we state in the following result.

Proposition II.3.3. *Let us consider the operator* $\mathcal{T}_\varepsilon$ *defined by* (II.23). *The associated dissipation, defined bellow, satisfies*

$$
\mathcal{D}_\varepsilon(f) := -\iint \mathcal{T}_\varepsilon(f)\frac{f}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x = \iiint \left(\frac{f(v)}{F_\varepsilon(v)} - \frac{f(w)}{F_\varepsilon(w)}\right)^2 \frac{F_\varepsilon(v)}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x, \tag{II.28}
$$

and if we write $\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v$, *then for all* $f \in L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d\times\mathbb{R}^d)$ *we have*

$$
\mathcal{D}_\varepsilon(f) \geq \int (f - \rho F_\varepsilon)^2 \frac{\mathrm{d}x\,\mathrm{d}v}{F_\varepsilon(v)}. \tag{II.29}
$$

Proof. The Poincaré type inequality (II.29) is a particular case of the so-called Φ-entropy inequalites introduced in [GI08]. For the sake of completeness we shall give a sketch of the proof adapted to the case that we need.
We shall first start proving (II.28). Writing $\Phi_\varepsilon = v - \varepsilon^{2s-1}E(t,x)$ and $g = f/F_\varepsilon$, and since F_ε satisfies (II.24) we have:

$$
\begin{aligned}
\mathcal{D}_\varepsilon(f) &= -\iint \Big(\nabla_v\cdot(\Phi_\varepsilon g F_\varepsilon)\,g - \big(-\Delta_v\big)^s\,(gF_\varepsilon)\,g\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= -\iint \Big(\Phi_\varepsilon F_\varepsilon \frac{1}{2}\nabla_v(g^2) + \nabla_v\cdot(\Phi_\varepsilon F_\varepsilon)g^2 - \big(-\Delta_v\big)^s(g)gF_\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \Big(\frac{1}{2}g^2\big(-\Delta_v\big)^s(F_\varepsilon) - g^2\big(-\Delta_v\big)^s(F_\varepsilon) + g\big(-\Delta_v\big)^s(g)F_\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \Big(g\big(-\Delta_v\big)^s(g) - \frac{1}{2}\big(-\Delta_v\big)^s(g^2)\Big)F_\varepsilon\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

Hence, using (II.5) we see that:

$$
\begin{aligned}
&\iint \Big(g\big(-\Delta_v\big)^s(g) - \frac{1}{2}\big(-\Delta_v\big)^s(g^2)\Big) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \\
&\quad = \iiint \left(\frac{g(v)\big(g(v)-g(w)\big)}{|v-w|^{d+2s}} - \frac{1}{2}\frac{g^2(v)-g^2(w)}{|v-w|^{d+2s}}\right) F_\varepsilon(t,x,v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x \\
&\quad = \frac{1}{2}\iiint \frac{\big(g(v)-g(w)\big)^2}{|v-w|^{d+2s}} F_\varepsilon(t,x,v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

Recall that $F_\varepsilon(t,x,v) = F\big(v - \varepsilon^{2s-1}E(t,x)\big)$, therefore through a simple change of variable, if we call $h(t,x,v) = g\big(v - \varepsilon^{2s-1}E(t,x)\big)$ we have:

$$
\mathcal{D}_\varepsilon(f) = \frac{1}{2}\iiint \frac{\big(h(t,x,v)-h(t,x,w)\big)^2}{|v-w|^{d+2s}} F(v)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}x.
$$

In order to prove the control (II.29) we consider the semigroup associated with $\big(-\Delta\big)^s$

$$
\frac{\mathrm{d}}{\mathrm{d}t} P_t(h)(v) = -\big(-\Delta\big)^s\Big(P_t(h)\Big)(v) \tag{II.30}
$$

with $P_0(h)(v) = h(v)$ and we see, using (II.25), that if we introduce the kernel

$$
K_t(v) = \mathcal{F}^{-1}\Big(\kappa e^{-t|\xi|^{2s}/2s}\Big)(v)
$$

where κ is a constant normalizing K_1, then we have explicitly $P_t(h) = K_t * h$. For $s \in [0,t]$ we consider

$$
\psi(s) = P_s(H^2)(v) \tag{II.31}
$$

with $H = P_{t-s}(h)$. We then have for $s \in [0,t]$:

$$
\begin{aligned}
\psi'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[K_s * \Big(K_{t-s} * h\Big)^2\right] \\
&= \Big(\frac{\mathrm{d}}{\mathrm{d}s}K_s\Big) * \Big(K_{t-s} * h\Big)^2 + K_s * \frac{\mathrm{d}}{\mathrm{d}s}\Big[\big(K_{t-s} * h\big)^2\Big] \\
&= P_s\Big(-\big(-\Delta\big)^s H^2\Big) + 2P_s\Big(H\big(-\Delta\big)^s H\Big) \\
&= P_s\bigg(\int \frac{\big(H(v)-H(w)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\bigg).
\end{aligned}
$$

Using the integral expression of the convolution and Jensen's inequality it is straightforward to see that $\big(P_{t-s}(h)(v) - P_{t-s}(h)(w)\big)^2 \leq P_{t-s}\big(h(v)-h(w)\big)^2$. Therefore, using

Fubini's theorem, we have:

$$\psi'(s)(v) \le P_s\Big(P_{t-s}\Big(\int \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\Big)\Big) = P_t\Big(\int \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\Big).$$

Integrating over $s \in [0,t]$ one gets

$$P_t\big(h^2\big)(v) - \Big(P_t(h)(v)\Big)^2 \le tP_t\Big(\int \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+2s}}\,\mathrm{d}w\Big).$$

Finally, taking $t=1$ and evaluating at $v=0$ we get:

$$\int h^2(w)F(w)\,\mathrm{d}w - \Big(\int h(w)F(w)\,\mathrm{d}w\Big)^2 \le \iint \frac{\big(h(v)-h(w)\big)^2}{|v-w|^{d+2s}}F(v)\,\mathrm{d}v\,\mathrm{d}w. \quad \text{(II.32)}$$

Through a simple change of variables, inverse of the one we did earlier, we obtain

$$\int g^2(w)F_\varepsilon(w)\,\mathrm{d}w - \Big(\int g(w)F_\varepsilon(w)\,\mathrm{d}w\Big)^2 \le \iint \frac{\big(g(v)-g(w)\big)^2}{|v-w|^{d+2s}}F_\varepsilon(v)\,\mathrm{d}v\,\mathrm{d}w. \quad \text{(II.33)}$$

Finally, replacing g by f/F_ε, since F_ε is normalized, we recover (II.29).
□

Since the operator $\mathcal{T}_\varepsilon$ is negative semidefinite in $L^2_{F_\varepsilon^{-1}}(\mathbb{R}^d)$ it is natural to look for bounds of the quadratic entropy associated to solutions f_ε of (II.4). We gather the appropriate a priori estimates that we shall need to pass to the limit in (II.4) in the following Proposition.

Proposition II.3.4. *Let the assumptions of Theorem II.1.4 be satisfied and let f_ε be the solution of* (II.4). *We introduce the residue r_ε through the macro-micro decomposition $f_\varepsilon = \rho_\varepsilon F_\varepsilon + \varepsilon^s r_\varepsilon$. Then, uniformly in $\varepsilon \in (0,1)$, we have:*

(i) *(f_ε) is bounded in $L^\infty([0,T); L^2_{F^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$ and in $L^\infty([0,T); L^1(\mathbb{R}^d \times \mathbb{R}^d))$,*

(ii) *(ρ_ε) is bounded in $L^\infty([0,T); L^2(\mathbb{R}^d))$,*

(iii) *(r_ε) is bounded in $L^2([0,T); L^2_{F^{-1}(v)}(\mathbb{R}^d \times \mathbb{R}^d))$.*

Proof. Multiplying (II.4) by $f_\varepsilon/F_\varepsilon$, integrations by parts yield

$$\frac{\varepsilon^{2s-1}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x+\frac{\varepsilon^{2s-1}}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\frac{\partial_t F_\varepsilon}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x$$
$$-\frac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\frac{v\cdot\nabla_x F_\varepsilon}{F_\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{\varepsilon}\mathcal{D}_\varepsilon(f^\varepsilon)=0.$$

Thus, thanks to Proposition II.3.2, part (i) and (ii), and (II.29) we obtain

$$\frac{\varepsilon^{2s}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x+\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(f_\varepsilon-\rho_\varepsilon F_\varepsilon)^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\le\varepsilon^{2s}\mu\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x. \tag{II.34}$$

Whence, part (i) follows by Gronwall's lemma and the fact that the weights $1/F$ and $1/F_\varepsilon$ are equivalent uniformly in ε which follows from Proposition II.3.2, part (i). On the other hand, part (ii) follows thanks to the inequality

$$\rho_\varepsilon\le\left(\int\frac{f_\varepsilon^2}{F_\varepsilon}\,\mathrm{d}v\right)^{1/2},$$

which is an immediate consequence of Cauchy-Schwarz and the fact $\int F_\varepsilon\,\mathrm{d}v=1$. Finally, part (iii) follows from (II.48) after integrating with respect to t over $(0,T)$ and thanks to Proposition II.3.2 part (ii).

□

II.4 Anomalous diffusion limit

We shall follow the method introduced in [CMT12]. Let us start by introducing the following auxiliary problem: for $\psi\in\mathcal{C}_c^\infty([0,T)\times\mathbb{R}^d)$, define ϕ_ε the unique solution of

$$\begin{aligned}&\varepsilon v\cdot\nabla_x\phi_\varepsilon-v\cdot\nabla_v\phi_\varepsilon=0 &&\text{in } [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d,\\ &\phi_\varepsilon(t,x,0)=\psi(t,x) &&\text{in } [0,\infty)\times\mathbb{R}^d\end{aligned} \tag{II.35}$$

The function ϕ_ε can be obtained readily via the method of characteristics and can be expressed in an explicit manner as follows:

$$\phi_\varepsilon(t,x,v)=\psi(t,x+\varepsilon v). \tag{II.36}$$

Next, since ϕ_ε is in $\mathcal{C}_c^\infty(Q_T)$ we can use it as a test function in (II.14) and we obtain

$$\iiint_{Q_T} f_\varepsilon\Big(\varepsilon^{2s-1}\,\partial_t\phi_\varepsilon + v\cdot\nabla_x\phi_\varepsilon - \frac{1}{\varepsilon}(v-\varepsilon^{2s-1}E)\cdot\nabla_v\phi_\varepsilon - \frac{1}{\varepsilon}\big(-\Delta_v\big)^s\phi_\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\,\varepsilon^{2s-1}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\phi_\varepsilon(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0\,. \qquad \text{(II.37)}$$

Let us note the following

$$\big(-\Delta_v\big)^s\phi_\varepsilon(t,x,v) = \varepsilon^{2s}\big(-\Delta\big)^s\psi(t,x+\varepsilon v), \qquad \text{(II.38)}$$
$$\nabla_v\phi_\varepsilon(t,x,v) = \varepsilon\nabla\psi(t,x+\varepsilon v), \qquad \text{(II.39)}$$

which follows after a simple computation using the definition (II.5) of the fractional Laplacian. Thus using the auxiliary equation (II.35) and plugging (II.38) into (II.37) yields

$$\int_0^\infty\iint f_\varepsilon\Big(\partial_t\psi(t,x+\varepsilon v) + E\cdot\nabla_x\psi(t,x+\varepsilon v) - \big(-\Delta\big)^s\psi(t,x+\varepsilon v)\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\iint f^{in}(x,v)\psi(0,x+\varepsilon v)\,\mathrm{d}v\,\mathrm{d}x = 0\,. \qquad \text{(II.40)}$$

II.4.1 The non-critical case: $1/2 < s < 1$

In order to pass to the limit in this weak formulation, we introduce the following two results.

Lemma II.4.1. *Let (f_ε) be the sequence of solutions of* (II.4)*, and ρ be the limit of (ρ_ε) which exists thanks to Proposition II.3.4 part (ii), then*

$$f_\varepsilon(t,x,v) \rightharpoonup \rho(t,x)F(v) \quad \textit{weakly in } L^\infty([0,T];L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))$$

Proof. This lemma follows directly from Proposition II.3.4. Since f_ε is uniformly bounded, it converges weakly in $L^\infty([0,T];L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))$. From the bounds on F_ε established in Proposition II.3.2 and the boundedness of ρ_ε in $L^\infty([0,T];L^2(\mathbb{R}^d))$ we see that $\rho_\varepsilon(t,x)F_\varepsilon(v)$ converges to $\rho(t,x)F(v)$ weakly in $L^\infty([0,T];L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))$ where ρ is the weak limit of ρ_ε. Finally, since the residue r_ε is bounded, it follows from the micro-macro decomposition $f_\varepsilon = \rho_\varepsilon F_\varepsilon + \varepsilon^s r_\varepsilon$ that the limit of f_ε is the same as the limit of $\rho_\varepsilon F_\varepsilon$.

□

Lemma II.4.2. *For all test functions ψ in $C_c^\infty([0,\infty)\times\mathbb{R}^d)$ we have:*

$$\lim_{\varepsilon\to 0}\iiint\limits_{Q_T} f^\varepsilon(t,x,v)\psi(t,x+\varepsilon v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iint\limits_{[0,T)\times\mathbb{R}^d} \rho(t,x)\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{II.41}$$

Moreover, if $E(t,x)\in\left(W^{1,\infty}([0,T)\times\mathbb{R}^d)\right)^d$ then for all $\Psi\in C_c^\infty([0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$ the following convergence holds:

$$\lim_{\varepsilon\to 0}\iiint\limits_{Q_T} f^\varepsilon(t,x,v)E(t,x)\cdot\Psi(t,x+\varepsilon v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iint\limits_{[0,T)\times\mathbb{R}^d} \rho(t,x)E(t,x)\cdot\Psi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{II.42}$$

Proof. We will give a detailed proof of the convergence in (II.42), the convergence in (II.41) follows as a consequence of (II.42) by taking $\psi(t,x+\varepsilon v)=E(t,x)\cdot\Psi(t,x+\varepsilon v)$, with a smooth E, and Lemma II.4.1. For (II.42), we write:

$$\begin{aligned}\iiint\limits_{Q_T} f_\varepsilon E(t,x)\cdot\Psi(t,x+\varepsilon v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t &= \iint\limits_{[0,T)\times\mathbb{R}^d}\rho(t,x)E(t,x)\cdot\Psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\ &+\iiint\limits_{Q_T}\Big(f_\varepsilon-\rho(t,x)F(v)\Big)E(t,x)\cdot\Psi(t,x)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &+\iiint\limits_{Q_T} f_\varepsilon E(t,x)\cdot\Big(\Psi(t,x+\varepsilon v)-\Psi(t,x)\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{II.43}$$

The second term in the right hand side of (II.43) converges to zero since f_ε converges to ρF weakly in $L^\infty([0,T);L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))$ thanks to Lemma II.4.1. For the third term on the right hand side of (II.43) thanks to Cauchy-Schwarz and Hölder we obtain

$$\begin{aligned}&\left|\iiint\limits_{Q_T} f_\varepsilon E(t,x)\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\le\int_0^T\left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f_\varepsilon^2}{F}\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}\Big[E(t,x)\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\Big]^2F\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\mathrm{d}t\\ &\le\|f_\varepsilon\|_{L^\infty([0,T);L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d))}\\ &\qquad\times\int_0^T\left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d}[E(t,x)\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))]^2F\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\mathrm{d}t.\end{aligned} \tag{II.44}$$

Next, let R be an arbitrary positive real number and let us consider the following splitting

$$
\begin{aligned}
\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} & \Big[E\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\Big]^2 F(v)\,\mathrm{d}v\,\mathrm{d}x \\
&= \int\limits_{\mathbb{R}^d}\int\limits_{|v|\le R} \Big[E\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\Big]^2 F(v)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad + \int\limits_{\mathbb{R}^d}\int\limits_{|v|> R} \Big[E\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\Big]^2 F(v)\,\mathrm{d}v\,\mathrm{d}x. \qquad \text{(II.45)}
\end{aligned}
$$

We will use the regularity of Ψ to bound the integral on $|v|<R$. To that end, let us consider the εR neighborhood of the support of Ψ denoted as $\Omega(\varepsilon R)$ which consists of the union of all the balls of radius εR having as center a point in $\operatorname{supp}\Psi$. Next, let Λ denote the diameter of $\operatorname{supp}\Psi$ defined as the maximum over all the distances between two points in $\operatorname{supp}\Psi$. Then it is clear that $\Omega(\varepsilon R)\subseteq B(x_0;\Lambda+\varepsilon R)$ where $B(x_0;\Lambda+\varepsilon R)$ denotes the ball with center at x_0 and radius $\Lambda+\varepsilon R$ and x_0 is any arbitrary fix point in $\operatorname{supp}\Psi$. Then for the integral over $|v|<R$ we have the following

$$
\begin{aligned}
\int\limits_{\mathbb{R}^d}\int\limits_{|v|\le R} & [E\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))]^2 F(v)\,\mathrm{d}v\,\mathrm{d}x \\
&\le \|F\|_{L^\infty(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d}\int\limits_{|v|\le R} \left(\sum_{j=1}^d |E_j|\big|\varepsilon v\cdot\nabla_x\Psi_j(t,x+\theta_j\varepsilon v)\big|\right)^2 \mathrm{d}v\,\mathrm{d}x \\
&\le 2\varepsilon^2\|F\|_{L^\infty(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d}\int\limits_{|v|\le R} |v|^2\left(\sum_{j=1}^d |E_j|^2\big|\nabla_x\Psi_j(t,x+\theta_j\varepsilon v)\big|^2\right) \mathrm{d}v\,\mathrm{d}x \\
&\le 2\varepsilon^2\|F\|_{L^\infty(\mathbb{R}^d)}\|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)}\|\nabla_x\Psi\|_{L^\infty(\mathbb{R}^d)} \int_{|v|\le R}\int_{B(x_0,\delta+\varepsilon R)} |v|^2\,\mathrm{d}x\,\mathrm{d}v \\
&\le \varepsilon^2 C_2(\Lambda+\varepsilon R)^d R^{d+2}, \qquad \text{(II.46)}
\end{aligned}
$$

where C_2 is a constant depending on $\|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)}$, $\|F\|_{L^\infty(\mathbb{R}^d)}$ and $\|D_x^2\phi\|_{L^\infty(\mathbb{R}^d)}$ but not on ε, and $\theta_j\in(0,1)$ for $j=1,\dots,d$ is such that $\Psi_j(t,x+\varepsilon v)-\Psi_j(t,x)=\varepsilon v\cdot\nabla_x\Psi_j(t,x+\theta_j\varepsilon v)$. For the integral on $|v|>R$ we use the decay of the equilibrium

$F(v)$ to derive the following upper bound:

$$\int_{\mathbb{R}^d}\int_{|v|>R}\Big[E\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\Big]^2 F(v)\,\mathrm{d}v\,\mathrm{d}x$$

$$\begin{aligned}
&\le \|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)}\int_{|v|>R}\left(\int_{\mathbb{R}^d}\Big(2|\Psi(t,x+\varepsilon v)|^2+2|\Psi(t,x)|^2\Big)\,\mathrm{d}x\right)F(v)\,\mathrm{d}v\\
&\le 4\|E\|^2_{W^{1,\infty}([0,T)\times\mathbb{R}^d)}\int_{\mathbb{R}^d}|\Psi(t,x)|^2\,\mathrm{d}x\int_{|v|>R}F(v)\,\mathrm{d}v\\
&\le C\int_{|v|>R}F(v)\,\mathrm{d}v.
\end{aligned}$$

Thanks to Proposition II.3.1, for any $\eta>0$ we can choose $R>0$ big enough such that

$$\left|F(v)-\frac{\vartheta}{|v|^{d+2s}}\right|\le\frac{\eta}{|v|^{d+2s}},\qquad\text{for all } |v|\ge R.$$

Thus choosing $\eta=\vartheta$ we have the following estimate:

$$\begin{aligned}
\int_{|v|>R}F(v)\,\mathrm{d}v&\le\int_{|v|>R}\left|F(v)-\frac{\vartheta}{|v|^{d+2s}}\right|\mathrm{d}v+\int_{|v|>R}\frac{\vartheta}{|v|^{d+2s}}\,\mathrm{d}v\\
&\le 2\int_{|v|>R}\frac{\vartheta}{|v|^{d+2s}}\,\mathrm{d}v\\
&\le\frac{C}{R^{2s}}.
\end{aligned}$$

From which we conclude

$$\int_{\mathbb{R}^d}\int_{|v|>R}\Big[E\cdot(\Psi(t,x+\varepsilon v)-\Psi(t,x))\Big]^2F(v)\,\mathrm{d}v\,\mathrm{d}x\le\frac{C_2}{R^{2s}}. \tag{II.47}$$

Next let us note that for any $\delta>0$ we can choose $\widetilde{R}>0$ such that $C_2/R^{2s}<\delta/2$ for all $R>\widetilde{R}$ and then choose $\varepsilon>0$ so that $\varepsilon^2C_1(\Lambda+\varepsilon R)^dR^{d+2}<\delta/2$. And thus deduce that for ε small enough we have

$$\varepsilon^2C_1(\Lambda+\varepsilon R)^dR^{d+2}+\frac{C_2}{R^{2s}}<\delta.$$

Therefore, plugging (II.46) and (II.47) into (II.44) and using Proposition II.3.4, part (i), we obtain that there exists a fixed $C > 0$ such that

$$\begin{aligned}\Big|\iiint_{Q_T} f_\varepsilon E\cdot\big(\Psi(t,x+\varepsilon v)-\Psi(t,x)\big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\Big| \\ &\le C\Big(\varepsilon^2 C_1(\Lambda+\varepsilon R)^d R^{d+2}+\frac{C_2}{R^{2s}}\Big) \\ &\le C\delta,\end{aligned}$$

for any $\delta > 0$, hence concluding that the third term on the right hand side of (II.43) goes to zero as $\varepsilon \to 0$. □

Using Lemma II.4.2 we can now take the limit in (II.40) and conclude that ρ satisfies

$$\iint_{[0,T)\times\mathbb{R}^d} \rho\Big(\partial_t\phi + E\cdot\nabla_x\phi - \big(-\Delta_x\big)^s\phi\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \rho_{in}(x)\phi(0,x)\,\mathrm{d}x = 0,$$

for all $\phi \in C_c^\infty([0,T)\times\mathbb{R}^d)$. Thus concluding the proof of Theorem II.1.4.

II.4.2 The critical cases $s = 1/2$ and $s = 1$

In the critical case $s = 1$ we recover the classical Fokker-Planck operator which means, in particular, as mentioned in the Introduction, that its equilibrium is a Maxwellian $M(v) = C\exp\left(-|v|^2\right)$ instead of the heavy-tail distribution F. We can still consider the perturbed operator $\mathcal{T}_\varepsilon$ of Proposition II.3.1 and its equilibrium will also be a translation of the unperturbed one:

$$F_\varepsilon(t,x,v) = Ce^{-|v-\varepsilon E(t,x)|^2}$$

and since the decay of the Maxwellian is much faster than the decay of the heavy-tail distributions, Proposition II.3.2 holds. The dissipative properties of the Fokker-Planck operator are well known, see e.g. [CH16] [GNPS05] or [BG08], and it is straightforward to check the boundedness results of Proposition II.3.4. Hence, Lemma II.4.1 holds and we can take the limit in the weak formulation (II.40) to prove that Theorem II.1.4 holds in the case $s = 1$.

In the critical case $s = 1/2$, the perturbed operator $\mathcal{T}_\varepsilon$ of (II.23) and its equilibrium F_ε (II.26) lose their dependence with respect to ε:

$$\mathcal{T}_\varepsilon(f_\varepsilon) = \mathcal{T}_E(f_\varepsilon) := \nabla_v \cdot \Big[\big(v - E(t,x)\big) f_\varepsilon\Big] - \big(-\Delta_v\big)^s f_\varepsilon,$$
$$F_\varepsilon(t,x,v) = F_E(t,x,v) := F\big(v - E(t,x)\big)$$

where F is the normalized equilibrium of $\mathcal{L}^{1/2}$. In particular, the equilibrium F_E will remain unchanged in the limit as ε goes to 0 and Proposition II.3.2 will hold with $s = 1/2$ which, in particular, means that the bounds in (*ii*) and (*iii*) do not go to zero. The operator is still dissipative since the dependence on ε does not matter in the proof of Proposition II.3.3, hence we still have (II.33) and multiplying (II.4) by f_ε/F_E and integrating by parts yields:

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f_\varepsilon^2}{F_E}\,\mathrm{d}v\,\mathrm{d}x + \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f_\varepsilon - \rho_\varepsilon F_E)^2}{F_E}\,\mathrm{d}v\,\mathrm{d}x \le \varepsilon\mu \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f_\varepsilon^2}{F_E}\,\mathrm{d}v\,\mathrm{d}x. \tag{II.48}$$

Since E is in $\big(W^{1,\infty}([0,T)\times\mathbb{R}^d)\big)^d$, if $f_\varepsilon(t,\cdot,\cdot)$ is in $L^2_{F_E(t,x,v)}(\mathbb{R}^d\times\mathbb{R}^d)$ and bounded independently of time, then it is also in $L^2_{F(v)}(\mathbb{R}^d\times\mathbb{R}^d)$. As a consequence, from (II.48) we still have the uniform in ε boundedness of f_ε, $\rho_\varepsilon = \int f_\varepsilon\,\mathrm{d}v$ and the residue r_ε in $L^\infty([0,T); L^2_{F(v)}(\mathbb{R}^d\times\mathbb{R}^d))$ as stated in Proposition II.3.4. This yields the following modified version of Lemma 1:

Lemma II.4.3. *Let $s = 1/2$, (f_ε) be the sequence of solutions of* (II.4)*, and ρ be the limit of (ρ_ε) which exists thanks to Proposition II.3.4 part (ii), then*

$$f_\varepsilon(t,x,v) \rightharpoonup^\star \rho(t,x) F_E(t,x,v) \quad \text{in } L^\infty([0,T); L^2_{F^{-1}(v)}(\mathbb{R}^d\times\mathbb{R}^d)).$$

Finally, for the proof of convergence of the weak formulation (II.40), i.e. the proof of Lemma II.4.2, we proceed essentially the same way. The only slight difference is that in order to control the third term of (II.43) we will use Cauchy-Schwarz as in (II.44) but we multiply and divide by $F(v)^{1/2}$ instead of the natural equilibrium F_E. The rest of the proof remains the same and we can then take the limit in the weak formulation, which concludes the proof of Theorem II.1.4 with $s = 1/2$.

Chapter III

Anomalous diffusion limit in spatially bounded domains

Contents

III.1 Introduction

Because of the non-local nature of fractional diffusion, it is not clear how it should interact with a boundary. In an effort to understand this interaction, we present in this chapter the derivation of fractional diffusion equations on spatially bounded domain from kinetic equations with a fractional Fokker-Planck collision operator. This setting is particularly relevant due the fact that, in those kinetic equations, the non-local collision operator acts solely on the velocities of the particles which are unbounded, hence the interaction between the non-local phenomena in position and the spatial boundary will only arise as we look at the anomalous diffusion limit.
We investigate the long time/small mean-free-path asymptotic behaviour of the solution of the fractional Vlasov-Fokker-Planck (VFP) equation:

$$\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(vf) - (-\Delta_v)^s f \qquad \text{in } \mathbb{R}^+\times\Omega\times\mathbb{R}^d, \tag{III.1a}$$

$$f(0,x,v) = f_{in}(x,v) \qquad \text{in } \Omega\times\mathbb{R}^d, \tag{III.1b}$$

for $s\in(0,1)$ on a smooth convex domain Ω. We introduce the oriented set:

$$\Sigma_\pm = \{(x,v)\in\Sigma;\, \pm n(x)\cdot v>0\} \text{ with } \Sigma = \partial\Omega\times\mathbb{R}^d \tag{III.2}$$

where $n(x)$ is the outgoing normal vector and we denote by γf the trace of f on $\mathbb{R}^+\times\partial\Omega\times\mathbb{R}^d$. The boundary conditions then take the form of a balance between the values of the traces of f on these oriented sets $\gamma_\pm f := \mathbb{1}_{\Sigma_\pm}\gamma f$. We will consider two types of conditions introduced by J. C. Maxwell in the appendix of [Max79] in 1879:

- The absorption boundary condition : for all $(x,v)\in\Sigma_-$

$$\gamma_- f(t,x,v) = 0 \tag{III.3}$$

- The local-in-velocity reflection operator called *specular reflection*: for all $(x, v) \in \Sigma_-$
$$\gamma_- f(t, x, v) = \gamma_+ f\big(t, x, \mathcal{R}_x(v)\big) \tag{III.4}$$
where $\mathcal{R}_x(v) = v - 2\big(n(x) \cdot v\big)n(x)$ which is illustrated in Figure III.1.

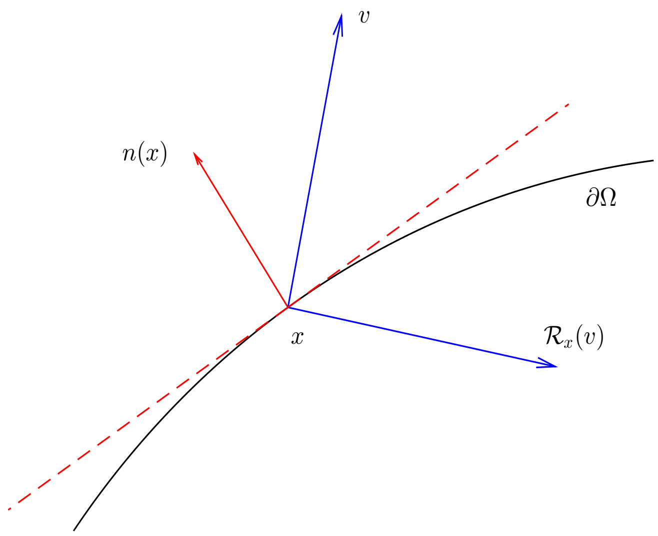


Fig. III.1 Specular reflection operator

The fractional VFP equation models the evolution of the distribution function $f(t, x, v)$ of a cloud of particles in a plasma. The left hand side of (III.1a) models the free transport of the particles, while on the right hand side the fractional Fokker-Planck operator
$$\mathcal{L}^s f = \nabla_v \cdot (vf) - (-\Delta_v)^s f \tag{III.5}$$
describes the interactions of the particles with the background. It can be interpreted as a deterministic description of a Langevin equation for the velocity of the particles, $\dot{v}(t) = -v(t) + A(t)$, where $A(t)$ is a white noise. This model describes the evolution of the velocity of a particle as the result of two phenomena, a viscosity-like interaction that causes the velocity to slow down and a white noise that causes it to jump at random times which can be interpreted as the consequence of collisions. The classical Fokker-Planck operator corresponds to $s = 1$ and arises when $A(t)$ is a Gaussian white noise. In that case, equilibrium distributions (solutions of $\mathcal{L}^1 M = 0$) are Maxwellian

(or Gaussian) velocity distributions $M = C\exp(-|v|^2/2)$. However, some experimental measurements of particles and heat fluxes in confined plasma point to non-local features and non-Gaussian distribution functions. The introduction of Lévy statistic in the velocity equation (replacing the Gaussian white noise by Lévy white noise in the Langevin equation) can be seen as an attempt at taking into account these non-local effects in plasma turbulence.
In order to study the long time/small mean free path asymptotic behaviour of the solutions of the fractional VFP equation, we introduce the Knudsen number ε which represents the ratio of the mean-free-path to the macroscopic length scale, or equivalently the ratio of the mean time between two collisions to the macroscopic time scale. We use this ε to rescale the time variable as

$$t' = \varepsilon^{2s-1}t. \tag{III.6}$$

Moreover, we also introduce $1/\varepsilon$ as a factor of the fractional Fokker-Planck operator to model the mean-free-path growing smaller as a consequence of the number of collisions per unit of time increasing. Hence, we consider the following scaling of (III.1a)-(III.1b):

$$\varepsilon^{2s-1}\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}\mathcal{L}^s\big(f^\varepsilon\big) \qquad \text{in } \mathbb{R}^+\times\Omega\times\mathbb{R}^d, \tag{III.7a}$$

$$f^\varepsilon(0,x,v) = f_{in}(x,v) \qquad \text{in } \Omega\times\mathbb{R}^d. \tag{III.7b}$$

and investigate the behaviour of the solution f^ε when ε goes to 0.

In the non-fractional framework, the diffusion limits under parabolic scaling of the Vlasov-Fokker-Planck equations, which is exactly (III.7a) with $s = 1$, have been established on the whole space in 2000 by Poupaud-Soler in [PS00] for a small enough time interval. They actually studied the more complicated Vlasov-Poisson-Fokker-Planck system but it is easy to see that their results imply, for the VFP equation, that the solution f^ε converges, as ε goes to 0, to $\rho(t,x)M(v)$ where M is the Maxwellian equilibrium of the Fokker-Planck operator and ρ is the limit of the density $\rho^\varepsilon = \int f^\varepsilon\,\mathrm{d}v$ and satisfies a Heat-equation. Their results were then extended (still in the Poisson case) in 2005 by Goudon [Gou05] to a global in time convergence in dimension 2 with bounds on the entropy and energy of the initial data as to ensure that they don't develop singularities in the limit system, and later in 2010 by El Ghani-Masmousi [EGM10] who proved the global in time convergence in higher dimensions with similar initial bounds.

In the fractional framework, Biler-Karch [BK03] and Gentil-Imbert [GI08] investigate the long-time behaviour of Lévy-Fokker-Planck equations

$$\partial_t f = \operatorname{div}(f\nabla\phi) + \mathcal{I}[f] \tag{III.8}$$

where $\mathcal{I}$ is the infinitesimal generator of a Lévy process. This family of operators includes the fractional Fokker-Planck operator since the fractional Laplacian of order s is the generator of a particular $2s$-stable Lévy process whose characteristic exponent is $|\xi|^{2s}$. Biler-Karch prove convergence of the solution of (III.8) to the unique normalised equilibrium of the Lévy-Fokker-Planck operator, later improved by Gentil-Imbert to exponential convergence in a weighted L^2 space where the weight is prescribed by the equilibrium. Their proofs use entropy production methods and a modified logarithmic Sobolev inequality which we will use later on to establish a priori estimates on the solutions of the fractional VFP equation in a similar weighted L^2 space. In [GM06] and references within, Guan-Ma give a description of this equilibrium and proofs that it is, in particular, heavy-tailed, as stated below in Proposition III.1.1.
This characterisation of the equilibrium of the fractional Fokker-Planck operator and the entropy production method allowed the author with A.Mellet and K. Trivisa to establish in [CMT12] the anomalous diffusion limit of the fractional VFP equation. More precisely, we proved the following result:

Theorem (Theorem 1.2 in [CMT12]). *Assume that $f_0 \in L^2_{F^{-1}}(\mathbb{R}^d \times \mathbb{R}^d)$ where $F(v)$ is the normalised heavy-tailed equilibrium of the fractional Fokker-Planck operator. Then, up to a subsequence, the solution f^ε of the rescaled fractional VFP equation on the whole space* (III.7a)-(III.7b) *converges weakly in $L^\infty(0,T;L^2_{F^{-1}}(\mathbb{R}^d \times \mathbb{R}^d))$, as $\varepsilon \to 0$ to $\rho(t,x)F(v)$ where $\rho(t,x)$ solves*

$$\partial_t \rho + (-\Delta_x)^s \rho = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^d \tag{III.9a}$$

$$\rho(0,x) = \rho_0(x) \qquad \text{in } \mathbb{R}^d \tag{III.9b}$$

with $\rho_0(x) = \displaystyle\int f_0(x,v)\,dv$.

We can see how this result compares to the aforementioned diffusion limit of the classical Vlasov-Fokker-Planck. However, the method used in [CMT12] to derive this asymptotic behaviour is quite different from what is done is the non-fractional case, and rests upon the particular structure of the fractional VFP equation. Indeed, and this will be essential for the rest of this paper, if we consider the Fourier transform of (III.1a) in x and v (respective Fourier variables k and ξ) on $\mathbb{R}^d \times \mathbb{R}^d$ we get the following PDE:

$$\partial_t \hat{f}(t,k,\xi) + (k-\xi)\nabla_\xi \hat{f}(t,k,\xi) = -|\xi|^{2s}\hat{f}(t,k,\xi). \tag{III.10}$$

This PDE is scalar-hyperbolic so if we follow well-chosen *characteristic lines*, it becomes an ODE which can be solved explicitly. The main idea of [CMT12] is to transpose these *characteristic lines* in a non-Fourier setting in order to derive fractional diffusion. The method presented in the present work consists in adapting these same *characteristic lines* to a bounded domain in order to handle the interaction with a boundary.
Kinetic equations on bounded domains, because of their obvious physical relevance, have always received a lot of attention. There have been many works concerning existence of global weak solutions on bounded domains with absorbing-type or reflection-type boundary conditions. We would like to mention the work of Carrillo [Car98] on the VPFP system, as well as the work of Mellet-Vasseur [MV07] for the VFP equation coupled to compressible Navier-Stokes via drag force, because their techniques could be applied to the fractional VFP equation with some adaptations to handle the non-local property of the diffusion operator and we will indeed follow the line of reasoning of [Car98] to prove well-posedness of the specular diffusion equation in section III.5.

Hydrodynamical and diffusion limits in bounded domains have also been the subject of many works. For instance, in 1987, Degond-Mas-Gallic [DMG87] established the first rigorous diffusion limit for the (classical) VFP equation in 1 dimension on a bounded domain. This result has been improved many times (cf. references within [WLL15b]), and in 2015 Wu-Lin-Lui proved in [WLL15b] that the diffusion limit of a VPFP system for multiple species charged particles with reflection boundary conditions is a Poisson-Nernst-Planck system with homogeneous Neumann boundary conditions. Other examples of macroscopic limits are the work Masmoudi-Saint-Raymond who, in 2003, showed in [MSR03] that the Boltzmann equation with Maxwell boundary conditions converges to the Stokes-Fourier system with Navier boundary conditions, or, more recently, the work of Jiang-Levermore-Masmoudi who established in [JLM09] the acoustic limit for DiPerna-Lions solutions and recovered impermeable boundaries for the acoustic system.

Before stating our main results, let us present properly the fractional Laplacian and give some well-known properties of this operator and the associated fractional Fokker-Planck operator.

III.1.1 Preliminaries on the fractional Fokker-Planck operator

The fractional Laplacian can be defined as a pseudo-differential operator of symbol $|\xi|^{2s}$ which can be written in Fourier transform as:

$$\mathcal{F}\Big[\big(-\Delta\big)^s f(\xi)\Big] = |\xi|^{2s}\mathcal{F}[f](\xi). \tag{III.11}$$

Much like the Laplace operator is the infinitesimal generator of a Brownian motion, the fractional Laplacian is the generator of a Lévy process. More precisely, it is the generator of a Lévy process V_t whose transition density $\rho(t, y-x)$ relative to the Lebesgue measure is given in Fourier by:

$$\int_{\mathbb{R}^d} e^{iv\cdot\xi}\rho(t,v)\,\mathrm{d}v = e^{-t|\xi|^{2s}}.$$

The fractional Laplacian can also be written as a singular integral, which will be most useful in the PDE framework:

$$\big(-\Delta\big)^s f(v) = c_{s,d}\mathrm{P.V.}\int_{\mathbb{R}^d}\frac{f(v)-f(w)}{|v-w|^{d+2s}}\,\mathrm{d}w \tag{III.12}$$

where $c_{s,d}$ is a constant depending on s and the dimension d given by:

$$c_{d,s} = \left(\int_{\mathbb{R}^d}\frac{1-\cos(\zeta_1)}{|\zeta|^{d+2s}}\,\mathrm{d}\zeta\right)^{-1}. \tag{III.13}$$

The properties of this operator have been studied in 2007 by Silvestre in [Sil07] and more recently by DiNezza-Palatucci-Valdinoci in [DPV12] where they focus on the link between $\big(-\Delta\big)^s$ and the fractional Sobolev spaces $H^s(\mathbb{R}^d)$.

The interaction between the non-locality of the fractional Laplacian and the boundary of a domain raises a lot of questions. In 2003, Bogdan-Burdzy-Chen introduced in [BBC03] the notion of reflected $2s$-stable processes, which are the restriction of a $2s$-stable process, such as V_t defined above, to a open set Ω in $\mathbb{R}^d$. In particular, they define the killed process, constructed by adding a coffin state ∂ to $\mathbb{R}^d$ and defining W_t, the killed process associated with V_t, as:

$$W_t(\omega) = \begin{vmatrix} V_t(\omega) \text{ for } t \leq t_\Omega(\omega) \\ \partial \text{ for } t > t_\Omega(\omega) \end{vmatrix}$$

where $t_\Omega = \inf\{t > 0 : V_t \notin \Omega\}$ is the first exit time. The Dirichlet form of this process on $L^2(\Omega, dx)$ is $(\mathcal{C}, \mathcal{F}^\Omega)$ defined as:

$$\mathcal{F}^\Omega = \left\{ f \in L^2(\mathbb{R}^d) : \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\big(f(x) - f(y)\big)^2}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y < \infty \text{ and } f = 0 \text{ q.e. on } \mathbb{R}^d \setminus \Omega \right\}$$

$$\mathcal{C}(f, g) = \frac{1}{2} c_{d,s} \iint\limits_{\Omega \times \Omega} \frac{\big(f(x) - f(y)\big)\big(g(x) - g(y)\big)}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y + \int\limits_{\Omega} f(x) g(x) \kappa_\Omega(x) \, \mathrm{d}x$$

where q.e. means quasi everywhere and κ_Ω is the density of the killing measure of W_t given by:

$$\kappa_\Omega(x) = c_{d,s} \int\limits_{\mathbb{R}^d \setminus \Omega} \frac{1}{|x-y|^{d+2s}} \, \mathrm{d}y.$$

They also define more general reflected processes by extending the lifetime of the process beyond t_Ω. The killed process has a direct link with the PDE approach to fractional Laplacian on bounded domain. Indeed, in 2014, Felsinger-Kassmann-Voigt considered in [FKV13], the Dirichlet problem for non-local operators which, in case of the fractional Laplacian, reads:

$$\begin{aligned} \big(-\Delta\big)^s f &= u \quad \text{in } \Omega \\ f &= g \quad \text{on } \mathbb{R}^d \setminus \Omega. \end{aligned} \tag{III.14}$$

They introduced the Hilbert space $H_\Omega\big(\mathbb{R}^d; \frac{1}{|x-y|^{d+2s}}\big)$, which is exactly the space $\mathcal{F}^\Omega$ defined above, provided with the norm $||f||_{L^2(\mathbb{R}^d)} + \mathcal{C}(f, f)$. They wrote a variational formulation of the Dirichel problem (III.14) in that Hilbert space and proved existence and uniqueness of solutions. Note that their results actually include a large family of non-local operators, we stated it here for the fractional Laplacian since it is the subject of this paper, but their work goes far beyond. For regularity results on the solutions of the homogeneous dirichlet problem with fractional Laplacian inside the domain and up to the boundary, we refer the reader to Grubb [Gru13] and Ros-Oton-Serra [ROS14]. The fractional Fokker-Planck operator $\mathcal{L}^s$ has been introduced as a generalization of the classical Fokker-Planck operator for general Lévy stable processes in 2000 by

Yanovsky-Chechkin-Schertzer-Tur [YCST00] and the following year it was derived from the wider class of non-linear Langevin-type equation driven by a Lévy stable noise by Schertzer-Larchevêque-Duan-Yanovsky-Lovejoy in [SLD+01]. In the present paper, the most crucial property of the fractional Fokker-Planck operator will be the fact that its thermodynamical equilibrium is a Lévy stable distribution i.e. a heavy-tailed distribution, instead of the Maxwellian distribution that arise in the non-fractional setting. The explicit solution in Fourier transform of the equation $\mathcal{L}^s F = 0$ yields the following result

Proposition III.1.1. *For $s \in (0,1)$ and $\nu > 0$, there exists a unique normalized equilibrium distribution function $F(v)$, solution of*

$$\mathcal{L}^s(F) = \nu \nabla_v \cdot (vF) - (-\Delta_v)^s F = 0, \qquad \int_{\mathbb{R}^d} F(v)\, dv = 1. \tag{III.15}$$

Furthermore, $F(v) > 0$ for all v, and F is a heavy-tailed distribution function satisfying

$$F(v) \sim \frac{C}{|v|^{d+2s}} \qquad \text{as } |v| \to \infty.$$

For a more detailed presentation of the equilibrium of $\mathcal{L}^s$ we refer the reader to [ASC16] and references within.

III.1.2 Main Results

Throughout this paper, for any $T > 0$ we write $Q_T = [0,T) \times \bar{\Omega} \times \mathbb{R}^d$ and $\Sigma = \partial\Omega \times \mathbb{R}^d$ as mentioned in (V.2). Also, we will write $L^p(\Sigma_\pm)$ the Lebesgue space associated with the norm:

$$\|\gamma_\pm f\|_{L^p(\Sigma_\pm)} = \left(\iint_{\Sigma_\pm} |\gamma_\pm f|^p \big(n(x)\cdot v\big)\, \mathrm{d}\sigma(x)\, \mathrm{d}v \right)^{1/p} \tag{III.16}$$

As usually in the framework of fractional Vlasov-Fokker-Planck equations we use the following definitions of weak solutions

Definition III.1.1. *We say that f is a weak solution of the fractional VFP equation with Dirichlet type boundary conditions* (III.1a)-(III.1b)-(III.3) *on $[0,T)$ if*

$$\begin{aligned} &f(t,x,v) \geq 0 \qquad \forall (t,x,v) \in [0,T) \times \Omega \times \mathbb{R}^d \\ &f \in L^2_{t,x} H^s_v(Q_T) = \left\{ f \in L^2(Q_T), \frac{f(t,x,v) - f(t,x,w)}{|v-w|^{\frac{d+2s}{2}}} \in L^2(Q_T \times \mathbb{R}^d) \right\} \end{aligned} \tag{III.17}$$

satisfies

$$\gamma_{\pm} f \in L^1\big(0,T; L^1(\Sigma_{\pm})\big), \quad \text{and } \gamma_- f = 0 \tag{III.18}$$

and (III.1a) *holds in the sense that for any* ϕ *such that*

$$\begin{aligned} &\phi \in C^\infty(Q_T) \qquad \phi(T,\cdot,\cdot) = 0 \\ &\gamma_+\phi(t,x,v) = 0 \qquad \forall (t,x,v) \in [0,T) \times \Sigma_+ \end{aligned} \tag{III.19}$$

we have:

$$\begin{aligned} &\iiint\limits_{Q_T} f\Big(\partial_t \phi + v\cdot\nabla_x\phi - v\cdot\nabla_v\phi - \big(-\Delta_v\big)^s\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\ &- \iint\limits_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \end{aligned} \tag{III.20}$$

In the case of specular reflection, it is well known that reflective boundaries are often responsible for a loss of regularity of the traces of f. Hence, we define the following notion of weak solutions:

Definition III.1.2. *We say that* f *is a weak solution of* (III.1a)-(III.1b)-(III.4) *on* $[0,T)$ *if*

$$\begin{aligned} &f(t,x,v) \geq 0 \qquad \forall (t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^d \\ &f \in L^2_{t,x}H^s_v(Q_T) = \left\{ f \in L^2(Q_T), \frac{f(t,x,v) - f(t,x,w)}{|v-w|^{\frac{d+2s}{2}}} \in L^2(Q_T \times \mathbb{R}^d) \right\} \end{aligned} \tag{III.21}$$

and (III.1a) *holds in the sense that for any* ϕ *such that:*

$$\begin{aligned} &\phi \in C^\infty(Q_T) \qquad \phi(T,\cdot,\cdot) = 0 \\ &\gamma_+\phi(t,x,v) = \gamma_-\phi\big(t,x,\mathcal{R}_x(v)\big) \qquad \forall (t,x,v) \in [0,T) \times \Sigma_+ \end{aligned} \tag{III.22}$$

we have:

$$\begin{aligned} &\iiint\limits_{(0,T)\times\Omega\times\mathbb{R}^d} f\Big(\partial_t \phi + v\cdot\nabla_x\phi - v\cdot\nabla_v\phi - \big(-\Delta_v\big)^s\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\ &\qquad - \iint\limits_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \end{aligned} \tag{III.23}$$

The existence and uniqueness of such weak solutions that satisfy appropriate estimates can be established through an adaptation of the methods of Carrillo in [Car98]

or Mellet-Vasseur [MV07] in order to handle the non-local property of the diffusion operator. We do not dwell on this issue since it is not the focus of this paper.
In the first part of this paper, section III.2, we establish a priori estimates on the weak solutions, in both the absorption and the specular reflection case, using the dissipative property of the fractional Fokker-Planck operator. We then use those estimates to prove convergence of the weak solution of the rescaled fractional VFP equation:

Proposition III.1.2. *Let f_{in} be in $L^2_{F^{-1}(v)}(\Omega \times \mathbb{R}^d)$ and s be in $(0,1)$. The weak solution f^ε of the rescaled fractional VFP equation* (III.7a)-(III.7b) *with absorption* (III.3) *or specular reflections* (III.4) *on the boundary satisfies*

$$f^\varepsilon(t,x,v) \rightharpoonup \rho(t,x)F(v) \text{ weakly in } L^\infty\big(0,T; L^2_{F^{-1}(v)}(\Omega \times \mathbb{R}^d)\big) \tag{III.24}$$

where $\rho(t,x)$ is the limit of the macroscopic densities $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon \,\mathrm{d}v$.

In sections 3 and 4, we establish the anomalous diffusion limits, i.e. we identify the limit ρ as solution of a diffusion equation. The main idea of these proofs is to take advantage of the aforementioned scalar-hyperbolic structure of the fractional VFP equation in Fourier space (III.10). To that end, we introduce an auxiliary problem whose purpose is to construct, from any test function $\psi(t,x)$, a function $\phi^\varepsilon(t,x,v)$ which will be constant along the *characteristic lines* of the fractional VFP equation modified to take into account the boundary conditions, and such that $\lim_{\varepsilon\searrow 0}\phi^\varepsilon(t,x,v) = \psi(t,x)$. For the absorption boundary condition, the auxiliary problem reads for $\psi \in \mathcal{D}([0,T)\times\Omega)$:

$$\varepsilon v\cdot\nabla_x\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon = 0 \qquad \forall (t,x,v)\in\mathbb{R}^+\times\Omega\times\mathbb{R}^d, \tag{III.25a}$$

$$\phi^\varepsilon(t,x,0) = \psi(t,x) \qquad \forall (t,x)\in\mathbb{R}^+\times\Omega, \tag{III.25b}$$

$$\gamma_+\phi^\varepsilon(t,x,v) = 0 \qquad \forall (t,x,v)\in\mathbb{R}^+\times\Sigma_+. \tag{III.25c}$$

We construct a solution of this problem and use it as a test function in the weak formulation of (III.7a)-(III.7b)-(III.3). We then show that we can take the limit in the weak formulation to prove:

Theorem III.1.3. *Assume that f_{in} is in $L^2_{F^{-1}(v)}(\Omega \times \mathbb{R}^d)$ and s is in $(0,1)$. Then the solution f^ε of* (III.7a)-(III.7b)-(III.3)*, converges weakly in the sense of Proposition III.1.2 to $\rho(t,x)F(v)$ where the extension of $\rho(t,x)$ by 0 outside of Ω is a weak solution* *of*

$$\partial_t \rho + \big(-\Delta\big)^s \rho = 0 \qquad (t,x) \in [0,T) \times \Omega \tag{III.26a}$$
$$\rho(x,0) = \rho_{in}(x) \qquad x \in \Omega \tag{III.26b}$$
$$\rho(t,x) = 0 \qquad t \in [0,T), x \in \mathbb{R}^d \setminus \Omega \tag{III.26c}$$

where $\rho_{in}(x) = \int f_{in}\,\mathrm{d}v$, *in the sense that for all* $\psi \in \mathcal{D}([0,T)\times\mathbb{R}^d)$ *compactly supported in* Ω*:*

$$\iint\limits_{(0,T)\times\mathbb{R}^d} \rho(t,x)\big(\partial_t\psi(t,x) - \big(-\Delta\big)^s\psi(t,x)\big)\,\mathrm{d}t\,\mathrm{d}x + \int\limits_{\mathbb{R}^d} \rho_{in}(x)\psi(0,x)\,\mathrm{d}x = 0. \tag{III.27}$$

In this macroscopic equation, the extension by 0 of the function ρ can be interpreted as an extension of (III.3), the homogeneous Dirichlet boundary condition in the kinetic equation, to the whole complementary of the domain Ω as a consequence of the non-local nature of the fractional Laplacian operator.
For the specular reflection boundary condition, if we want follow the *characteristic lines* as they reflect on the boundary, we need to reduce (when $s \geq 1/2$) the set of test functions to $\mathfrak{D}_T(\Omega)$ defined as:

$$\mathfrak{D}_T(\Omega) = \Big\{\psi \in \mathcal{C}^\infty([0,T)\times\bar{\Omega}) \text{ s.t. } \psi(T,\cdot) = 0 \text{ and } \forall x \in \partial\Omega : \nabla_x\psi(t,x)\cdot n(x) = 0\Big\}. \tag{III.28}$$

The auxiliary problem reads for $\psi \in \mathfrak{D}_T$:

$$\varepsilon v\cdot\nabla_x\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon = 0 \qquad \forall (t,x,v) \in \mathbb{R}^+\times\Omega\times\mathbb{R}^d, \tag{III.29a}$$
$$\phi^\varepsilon(t,x,0) = \psi(t,x) \qquad \forall (t,x) \in \mathbb{R}^+\times\Omega, \tag{III.29b}$$
$$\gamma_+\phi^\varepsilon(t,x,v) = \gamma_-\phi^\varepsilon\big(t,x,\mathcal{R}_x(v)\big) \qquad \forall (t,x,v) \in \mathbb{R}^+\times\Sigma_+. \tag{III.29c}$$

In order to construct a solution of this auxiliary problem we study geodesic trajectories in a Hamiltonian billiard. These trajectories are given by, parametrised with $s \in [0,\infty)$

$$\begin{cases} \dot{x}(s) = \varepsilon v(s) & x(0) = x^{in} \in \Omega, \\ \dot{v}(s) = -v(s) & v(0) = v^{in} \in \mathbb{R}^d, \\ \text{If } x(s) \in \partial\Omega \text{ then } v(s^+) = \mathcal{R}_{x(s)}(v(s^-)), & \end{cases} \tag{III.30}$$

as illustrated in Figure III.2 for example when Ω is a ball. We construct a function $\eta : \Omega\times\mathbb{R}^d \mapsto \bar{\Omega}$ that will be constant along those trajectories, defined as $\eta(x^{in}, v^{in}) = \lim_{s\to\infty} x(s)$ which obviously, strongly depends on the geometry of the domain and we will show that it is well defined when Ω is a half-space or a ball. This η function allows us to find a solution of the auxiliary problem:

Proposition III.1.4. *If Ω is either a half-space or smooth and strongly convex, then there exists a function $\eta : \Omega \times \mathbb{R}^d \to \bar{\Omega}$ such that*

$$\phi^\varepsilon(t,x,v) = \psi\big(t, \eta(x, \varepsilon v)\big) \tag{III.31}$$

is a solution of the auxiliary problem (III.29a)-(III.29b)-(III.29c).

Although the regularity of this η function is rather simple to study in the half-space, it is much harder to understand in the ball and we will devote Appendix A to this investigation. In fact, it is strongly linked with the free transport equation. Indeed, if we consider the following free transport equation in a ball with specular reflection on the boundary and a homogeneous-in-velocity initial condition:

$$\begin{aligned}
&\partial_t f + v\cdot\nabla_x f = 0 && (t,x,v) \in [0,T)\times\Omega\times\mathbb{R}^d\\
&f(0,x,v) = \psi(x) && (x,v)\in\Omega\times\mathbb{R}^d\\
&\gamma_- f(t,x,v) = \gamma_+ f(t,x,\mathcal{R}_x v) && (t,x,v)\in[0,T)\times\partial\Omega\times\{v : v\cdot n(x) < 0\}
\end{aligned}$$

then, using (III.29a)-(III.29b)-(III.29c) and Proposition III.1.4 we can show that a solution of this problem is

$$f(t,x,v) = \psi\big(\eta(x,-tv)\big).$$

As a consequence, the regularity properties of η we establish in Appendix A can also be interpreted as a propagation of regularity with respect to the velocity for the previous free transport equation. We are then able to establish the following anomalous diffusion limit.

Theorem III.1.5. *Let Ω be either a half-space or a ball in $\mathbb{R}^d$ and assume that f_{in} is in $L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)$ and s is in $(0,1)$. Then the solution f^ε of* (III.7a)-(III.7b)-(III.3)*, converges weakly in the sense of Proposition III.1.2 to $\rho(t,x)F(v)$ where $\rho(t,x)$ satisfies, for any $\psi\in\mathcal{C}^\infty([0,T)\times\bar{\Omega})$ if $s<1/2$ and any $\psi\in\mathfrak{D}_T(\Omega)$ if $s\geq 1/2$:*

$$\iint\limits_{(0,T)\times\Omega} \rho(t,x)\Big(\partial_t\psi(t,x) - (-\Delta)^s_{SR}\psi(t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x + \int\limits_{\Omega}\rho_{in}(x)\psi(0,x)\,\mathrm{d}x = 0. \tag{III.32}$$

where $\rho_{in}(x) = \int f_{in}\,\mathrm{d}v$ *and* $(-\Delta)^s_{SR}$ *is defined as:*

$$(-\Delta)^s_{SR}\psi(x) = c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\psi(x) - \psi\big(\eta(x,w)\big)}{|w|^{d+2s}}\,\mathrm{d}w \qquad \text{(III.33)}$$

This new operator, which we call *specular diffusion operator*, can be seen as a modified version of the fractional Laplacian where the particles can jump from a position x to a position y in Ω not only through a straight line but also through trajectories that are specularly reflected when they hit the boundary, and the probability of this jump is $1/|w|^{d+2s}$ where $|w|$ is the length of the trajectory. Note that when Ω is $\mathbb{R}^d$, by definition we have $\eta(x,w) = x + w$ so that $(-\Delta)^s_{\mathrm{SR}}$ coincides with the full fractional Laplacian $\big(-\Delta\big)^s$ on $\mathbb{R}^d$.
Theorem III.1.5 can also be proved when Ω is a strip $\{x = (x', x_d) \in \mathbb{R}^d : -1 < x_d < 1\}$ or a cube using arguments from the half-space case in order to handle locally the interaction with the boundary, and from the ball case to handle the multitude of reflections a trajectory in a strip or a cube may undergo in a finite time. Moreover, in order to extend this theorem to general smooth and strongly convex domains, one only needs to prove that the trajectories described by η in that domain satisfy appropriate controls, similar to the ones we state in Lemma III.4.2 in the case of the ball which we prove in Appendix A. The rest of the proof would remain the same.
Finally, in the last section of this paper, we focus on the macroscopic equation (III.57) which we name *specular diffusion equation*. First, we establish properties of the specular diffusion operator $(-\Delta)^s_{\mathrm{SR}}$. Namely, in the half-space we show that it can be written as a kernel operator with a symmetric kernel:

$$(-\Delta)^s_{\mathrm{SR}}\psi(x) = P.V. \int_{\Omega} \big(\psi(x) - \psi(y)\big) K_\Omega(x,y)\,\mathrm{d}y \qquad \text{with} \quad K_\Omega(x,y) = K_\Omega(y,x). \qquad \text{(III.34)}$$

and such that the kernel is $2s$-singular. Then, in both the half-space and the ball, we show that the operator is symmetric and admits a integration by parts formula. From this formula we derive a scalar product and defined the associated Hilbert space $\mathcal{H}^s_{\mathrm{SR}}(\Omega)$ in the spirit of the fractional Sobolev spaces in their relation with the fractional Laplacian operators as is presented for instance in [DPV12]. We conclude this paper by studying the specular diffusion equation in this setting:

Theorem III.1.6. *Let* Ω *be a half-space or a ball in* $\mathbb{R}^d$, u_{in} *be in* $L^2((0,T)\times\Omega)$ *and* s *be in* $(0,1)$. *For any* $T > 0$, *there exists a unique weak solution* $u \in L^2(0,T;\mathcal{H}^s_{SR}(\Omega))$

of

$$\partial_t u + (-\Delta)^s_{SR} u = 0 \qquad (t,x) \in [0,T) \times \Omega \tag{III.35a}$$
$$u(0,x) = u_{in}(x) \qquad x \in \Omega \tag{III.35b}$$

in the sense that for any $\psi \in \mathcal{C}^\infty([0,T)\times\bar{\Omega})$ *if* $s < 1/2$ *and any* $\psi \in \mathfrak{D}_T$ *is* $s \geq 1/2$*, u satisfies if* Ω *is a half-space:*

$$\begin{aligned}&\iint\limits_{(0,T)\times\Omega} u\partial_t\psi \,\mathrm{d}t\,\mathrm{d}x - \int\limits_{\Omega} u_{in}(x)\psi(0,x)\,\mathrm{d}x \\ &- \frac{1}{2}\iiint\limits_{(0,T)\times\Omega\times\Omega} \big(u(t,x)-u(t,y)\big)\big(\psi(t,x)-\psi(t,y)\big)K(x,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned} \tag{III.36}$$

and if Ω *is the unit ball*

$$\begin{aligned}&\iint\limits_{(0,T)\times\Omega} u\partial_t\psi \,\mathrm{d}t\,\mathrm{d}x - \int\limits_{\Omega} u_{in}(x)\psi(0,x)\,\mathrm{d}x \\ &- \frac{1}{2}\iiint\limits_{(0,T)\times\Omega\times\mathbb{R}^d} \big(u(t,x)-u\big(t,\eta(x,v)\big)\big)\big(\psi(t,x)-\psi\big(t,\eta(x,v)\big)\big)\frac{\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v}{|v|^{d+2s}} = 0.\end{aligned} \tag{III.37}$$

Moreover, if Ω *is a half-space or a ball, then the macroscopic density* ρ *who satisfies* (III.32) *for all* $\psi \in \mathcal{C}^\infty([0,T)\times\bar{\Omega})$ *is* $s < 1/2$ *and any* $\psi \in \mathfrak{D}_T(\Omega)$ *if* $s \geq 1/2$*, is the unique weak solution of* (III.35a)-(III.35b).

This theorem highlights the fact that the interaction with the boundary in (III.35a)-(III.35b) is contained in the definition of the diffusion operator $(-\Delta)^s_{\mathrm{SR}}$ since we don't need to add a boundary condition in order to have well-posedness.
Here again, although we only look at the half-space and the ball, other geometries can be handled by our method such as a strip or a cube for example. Furthermore, the only obstacle to considering more general domains lies in understanding the function η is those domains in order to establish the symmetry of the specular diffusion operator and estimates on its singularity.

III.2 A priori estimates

In order to study the asymptotic behaviour of the weak solution of (III.7a)-(III.7b) with (III.3) or (III.4) boundary condition, we need a priori estimates. Those estimates will rely on the following dissipation property of the fraction Fokker-Planck operator $\mathcal{L}^s$

Proposition III.2.1. *For all f smooth enough, if we define the dissipation as:*

$$\mathcal{D}^s(f) := -\int_{\mathbb{R}^d} \mathcal{L}^s(f)\frac{f}{F}\,\mathrm{d}v \tag{III.38}$$

then there exists $\theta > 0$ such that

$$\mathcal{D}^s(f) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big(f(v)-f(w)\big)^2}{|v-w|^{d+2s}}\frac{\mathrm{d}v\,\mathrm{d}w}{F(v)} \geq \theta\int_{\mathbb{R}^d}\big|f(v)-\rho F(v)\big|^2\frac{\mathrm{d}v}{F(v)} \tag{III.39}$$

where $\rho = \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v$. Note, in particular, that $\mathcal{D}^s(f) \geq 0$.

Proof. We introduce the notation $g = f/F(v)$ and notice by expending the divergence and integrating by parts that:

$$\int_{\mathbb{R}^d} \nabla_v\cdot(vFg)g\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^d}\nabla_v\cdot(vF)g^2\,\mathrm{d}v.$$

We recall that F satisfies $\mathcal{L}^s(F) = 0$, which means $\nabla_v\cdot(vF) = (-\Delta)^s(F)$. By symmetry of the fractional Laplacian and the previous remark we have:

$$\begin{aligned}
\mathcal{D}^s(f) &= -\int_{\mathbb{R}^d}\Big(\nabla_v\cdot(vgF)g - (-\Delta)^s(gF)g\Big)\,\mathrm{d}v\\
&= -\int_{\mathbb{R}^d}\Big(\nabla_v\cdot(vF)g^2/2 - (-\Delta)^s(gF)g\Big)\,\mathrm{d}v\\
&= -\int_{\mathbb{R}^d}\Big((-\Delta)^s(F)g^2/2 - (-\Delta)^s(gF)g\Big)\,\mathrm{d}v\\
&= \int_{\mathbb{R}^d}\Big(-\frac{1}{2}F(-\Delta)^s(g^2) + Fg(-\Delta)^s(g)\Big)\,\mathrm{d}v.
\end{aligned}$$

Inputting the definition (III.12) of the fractional Laplacian we get:

$$\begin{aligned}\mathcal{D}^s(f) &= c_s \int_{\mathbb{R}^d} P.V. \int_{\mathbb{R}^d} \left\{ -\frac{1}{2}[g(v)^2 - g(w)^2] + g(v)^2 - g(w)g(v) \right\} \frac{F(v)}{|v-w|^{d+2s}} \,\mathrm{d}w \,\mathrm{d}v \\ &= \frac{c_s}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} F(v) \frac{[g(v)-g(w)]^2}{|v-w|^{d+2s}} \,\mathrm{d}v \,\mathrm{d}w. \\ &= \frac{c_s}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{f(v)}{F(v)} - \frac{f(w)}{F(w)} \right)^2 \frac{F(v)}{|v-w|^{d+2s}} \,\mathrm{d}v \,\mathrm{d}w.\end{aligned}$$

Since v and w play the same role in the integral, we can write

$$\mathcal{D}^s(f) = \frac{c_s}{4} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left[\left(\frac{f(v)}{F(v)} - \frac{f(w)}{F(w)} \right)^2 F(v) + \left(\frac{f(v)}{F(v)} - \frac{f(w)}{F(w)} \right)^2 F(w) \right] \frac{\mathrm{d}v \,\mathrm{d}w}{|v-w|^{d+2s}}.$$

Expending the integrand and grouping the terms adequately, it is not difficult to show that:

$$\mathcal{D}^s(f) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big(f(v)-f(w)\big)^2}{|v-w|^{d+2s}} \frac{\mathrm{d}v \,\mathrm{d}w}{F(v)}. \tag{III.40}$$

Finally, the second inequality in (III.39) comes from the modified logarithmic Sobolev inequality of Gentil-Imbert (Theorem 3 in [GI08]) which we can use here because $F(v)$ is the infinitely divisible law associated with the Lévy measure $1/|v|^{d+2s}$. We refer the interested reader to [ASC16] for a proof of this functional inequality in the fractional Laplacian case. □

The dissipation property of $\mathcal{L}^s$ allows us to prove the following:

Proposition III.1.2. *Let f_{in} be in $L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)$ and s be in $(0,1)$. The weak solution f^ε of the rescaled fractional VFP equation* (III.7a)-(III.7b) *with absorption* (III.3) *or specular reflections* (III.4) *on the boundary satisfies*

$$f^\varepsilon(t,x,v) \rightharpoonup \rho(t,x)F(v) \text{ weakly in } L^\infty\big(0,T; L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)\big) \tag{III.24}$$

where $\rho(t,x)$ is the limit of the macroscopic densities $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon \,\mathrm{d}v$.

Proof. Multiplying (III.7a) by $f^\varepsilon/F(v)$ and integrating over x and v one gets, after integrations by parts, for the absorption boundary condition:

$$\varepsilon^{2s-1}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega\times\mathbb{R}^d}(f^\varepsilon)^2\frac{\mathrm{d}x\,\mathrm{d}v}{F(v)}+\iint\limits_{\Sigma_+}|\gamma_+f^\varepsilon|^2|n(x)\cdot v|\frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)}+\frac{1}{\varepsilon}\int\limits_\Omega\mathcal{D}^s(f^\varepsilon)\,\mathrm{d}x=0$$

and in the specular reflections case:

$$\varepsilon^{2s-1}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega\times\mathbb{R}^d}(f^\varepsilon)^2\frac{\mathrm{d}x\,\mathrm{d}v}{F(v)}+\frac{1}{\varepsilon}\int\limits_\Omega\mathcal{D}^s(f^\varepsilon)\,\mathrm{d}x=0.$$

In both cases, since the dissipation in non-negative, we see that $\frac{\mathrm{d}}{\mathrm{d}t}\|f^\varepsilon\|_{L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)}\leq 0$ so $f^\varepsilon(t,\cdot,\cdot)$ is bounded in $L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)$. Moreover, we have

$$\iint\limits_{(0,T)\times\Omega}\mathcal{D}^s(f^\varepsilon)\,\mathrm{d}t\,\mathrm{d}x\leq\varepsilon^{2s}\Big(\|f_{in}\|_{L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)}-\|f^\varepsilon(T,x,v)\|_{L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)}\Big)\underset{\varepsilon\to0}{\longrightarrow}0$$

and furthermore, by definition of ρ^ε, we see that

$$\rho^\varepsilon\leq\Big(\int\limits_{\mathbb{R}^d}(f^\varepsilon)^2\frac{\mathrm{d}v}{F(v)}\Big)^{1/2}\Big(\iint\limits_{\mathbb{R}^d}F(v)\,\mathrm{d}v\Big)^{1/2}=\|f^\varepsilon\|_{L^2_{F^{-1}(v)}(\mathbb{R}^d)}$$

so that ρ^ε is also bounded in $L^\infty(0,T;L^2(\Omega))$. The boundedness of f^ε in $L^\infty\big(0,T;L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)\big)$ gives us the existence of a weak limit $\bar{f}$. Since the dissipation goes to 0, (III.39) implies that the limit is in the kernel of the fractional Fokker-Planck operator, i.e. there exists a function ρ such that $\bar{f}(t,x,v)=\rho(t,x)F(v)$. And finally, the boundedness of ρ^ε gives us existence of a weak limit $\bar\rho$ and by uniqueness of the limit $\bar\rho=\rho$, which concludes the proof. □

III.3 Absorption in a smooth convex domain

We focus in this section on the absorption boundary condition (III.3) and show how we can easily adapt the method developed in [CMT12] for the anomalous diffusion limit of the fractional Vlasov-Fokker-Planck equation to this bounded domain case. According to Definition III.1.1, if f_ε is a weak solution of the rescaled equation (III.7a)-(III.7b) with absorption (III.3) on the boundary then for all ϕ satisfying (III.19) we have

$$\iiint\limits_{Q_T} f^\varepsilon \Big(\varepsilon^{2s-1}\partial_t\phi - \varepsilon^{-1}\big(-\Delta_v\big)^s\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \tag{III.41a}$$

$$+\iiint\limits_{Q_T} f^\varepsilon \Big(v\cdot\nabla_x\phi - \varepsilon^{-1}v\cdot\nabla_v\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \tag{III.41b}$$

$$+\,\varepsilon^{2s-1}\iint\limits_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{III.41c}$$

We recognize, in (III.41b), the *characteristic lines* of (III.10). In order to take advantage of the scalar-hyperbolic structure of (III.10) we want to consider test functions which are constant along those lines. This is the purpose of the auxiliary problem.

III.3.1 Auxiliary problem

In the absorption case, it is rather simple to adapt the auxiliary problem introduced in [CMT12] to the domain Ω. For any $\psi\in\mathcal{D}([0,T)\times\Omega)$ we introduce the auxiliary problem:

$$\varepsilon v\cdot\nabla_x\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon = 0 \qquad \forall(t,x,v)\in\mathbb{R}^+\times\Omega\times\mathbb{R}^d, \tag{III.25a}$$

$$\phi^\varepsilon(t,x,0) = \psi(t,x) \qquad \forall(t,x)\in\mathbb{R}^+\times\Omega, \tag{III.25b}$$

$$\gamma_+\phi^\varepsilon(t,x,v) = 0 \qquad \forall(t,x,v)\in\mathbb{R}^+\times\Sigma_+. \tag{III.25c}$$

Since the boundary condition (III.25c) is immediately compatible with the assumption of compactly support in Ω for the test function ψ, the construction of the solution ϕ_ε is rather straightforward:

Proposition III.3.1. *For any $\psi\in\mathcal{D}([0,T)\times\Omega)$, ϕ^ε defined as:*

$$\phi^\varepsilon(t,x,v) = \bar\psi(t,x+\varepsilon v)$$

where $\bar\psi$ is the extension of ψ by 0 outside Ω, is a solution of (III.25a)-(III.25b)-(III.25c).

Proof. The proof is almost immediate. For (III.25a) we write:

$$\begin{aligned}\varepsilon v\cdot\nabla_x\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon &= \varepsilon v\cdot\nabla_x[\bar\psi(t,x+\varepsilon v)] - v\cdot\nabla_v[\bar\psi(t,x+\varepsilon v)]\\ &= \varepsilon v\cdot\nabla\bar\psi(t,x+\varepsilon v) - \varepsilon v\cdot\nabla\bar\psi(t,x+\varepsilon v) = 0.\end{aligned}$$

Moreover, the definition of ϕ^ε ensures (III.25b) and, thanks to the compact support of ψ in Ω we also see that $\phi^\varepsilon(t,x,v) = 0$ for any $(x,v) \in \Sigma_+$ since it means that $x + \varepsilon v \notin \Omega$. □

For such a ϕ^ε we see that:

$$
\begin{aligned}
\big(-\Delta_v\big)^s \phi^\varepsilon(t,x,v) &= c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\phi^\varepsilon(t,x,v) - \phi^\varepsilon(t,x,w)}{|v-w|^{d+2s}}\,\mathrm{d}w \\
&= c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\bar\psi(t,x+\varepsilon v) - \bar\psi(t,x+\varepsilon w)}{|v-w|^{d+2s}}\,\mathrm{d}w \\
&= c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\bar\psi(t,x+\varepsilon v) - \bar\psi(t,w)}{\varepsilon^{-d-2s}|x+\varepsilon v - w|^{d+2s}}\varepsilon^{-d}\,\mathrm{d}w \\
&= \varepsilon^{2s}\big(-\Delta\big)^s \bar\psi(t,x+\varepsilon v)
\end{aligned}
\tag{III.43}
$$

so that the weak formulation (III.41a)-(III.41b)-(III.41c) becomes

$$
\iiint_{Q_T} f^\varepsilon\Big(\partial_t\bar\psi - \big(-\Delta\big)^s\bar\psi(t,x+\varepsilon v)\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v + \iint_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\bar\psi\big(0,x+\varepsilon v\big)\,\mathrm{d}x\,\mathrm{d}v = 0.
\tag{III.44}
$$

III.3.2 Macroscopic Limit

In Section III.2 we proved that f^ε converges weakly in $L^\infty\big(0,T;L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)\big)$. Hence, in order to pass to the limit in the weak formulation (III.44) we need to show that

$$
\partial_t\bar\psi(t,x+\varepsilon v) - \big(-\Delta\big)^s\bar\psi(t,x+\varepsilon v) \underset{\varepsilon\to0}{\longrightarrow} \partial_t\bar\psi(t,x) - \big(-\Delta\big)^s\bar\psi(t,x)
\tag{III.45}
$$

at least strongly in $L^\infty\big(0,T;L^2_{F(v)}(\Omega\times\mathbb{R}^d)\big)$. The proof of this convergence is rather similar to its equivalent in the unbounded case presented in [CMT12]. As a consequence we will not give any unnecessary details and instead we briefly recall the main arguments. First, we note that the continuity of $\bar\psi$ readily implies the convergence of the second term in (III.44):

$$
\iint_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\bar\psi\big(0,x+\varepsilon v\big)\,\mathrm{d}x\,\mathrm{d}v \underset{\varepsilon\to0}{\longrightarrow} \int_\Omega \rho_{in}(x)\bar\psi(0,x)\,\mathrm{d}x.
$$

Secondly, the strong convergence of (III.45) follows from the fact that if $\bar{\psi}$ is in $\mathcal{D}([0,T)\times\Omega)$ then

$$\partial_t\bar{\psi} \in \mathcal{D}([0,T)\times\Omega) \quad \text{and} \quad \big(-\Delta\big)^s\bar{\psi} \in \mathcal{D}([0,T)\times\mathbb{R}^d)\cap L^2([0,T)\times\mathbb{R}^d)$$

because the pseudo-differential operator $\big(-\Delta\big)^s$ can be defined as an operator from the Schwartz space to $L^2(\mathbb{R}^d)$, see e.g. Proposition 3.3 in [DPV12]. As a consequence, it is straightforward to use dominated convergence on both terms and prove the strong convergence of (III.45) in $L^\infty\big(0,T;L^2_{F(v)}(\Omega\times\mathbb{R}^d)\big)$, noticing that $\int F(v)\,\mathrm{d}v = 1$. Hence, we can take the limit in the weak formulation and find that ρ satisfies:

$$\iint\limits_{(0,T)\times\Omega} \rho(t,x)\big(\partial_t\psi(t,x) - \big(-\Delta\big)^s\psi(t,x)\big)\,\mathrm{d}t\,\mathrm{d}x + \int\limits_{\Omega}\rho_{in}(x)\psi(0,x)\,\mathrm{d}x = 0. \tag{III.46}$$

Since ρ is the limit of ρ^ε it is only defined on Ω. If we extend it by 0 on the complementary $\mathbb{R}^d\setminus\Omega$, then we can integrate over $\mathbb{R}^d$ instead of Ω and that concludes the proof of Theorem III.1.3.

III.4 Specular Reflection in a smooth strongly convex domain

We now turn to the more challenging case of the specular reflection boundary condition (III.4). From Definition III.1.2 we know that if f_ε is a weak solution of fractional Vlasov-Fokker-Planck equation with specular reflection on the boundary (III.7a)-(III.7b)-(III.4) then for any ϕ satisfying

$$\begin{aligned} &\phi \in C^\infty(Q_T) \qquad \phi(T,\cdot,\cdot) = 0 \\ &\gamma_+\phi(t,x,v) = \gamma_-\phi\big(t,x,\mathcal{R}_x(v)\big) \qquad \forall (t,x,v)\in[0,T)\times\Sigma_+ \end{aligned} \tag{III.22}$$

we have, analogously to the absorption case:

$$\iiint_{Q_T} f^\varepsilon\Big(\varepsilon^{2s-1}\partial_t\phi - \varepsilon^{-1}\big(-\Delta_v\big)^s\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \tag{III.41a}$$

$$+\iiint_{Q_T} f^\varepsilon\Big(v\cdot\nabla_x\phi - \varepsilon^{-1}v\cdot\nabla_v\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \tag{III.41b}$$

$$+\,\varepsilon^{2s-1}\iint_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{III.41c}$$

Once again, we would like to take advantage of the scalar-hyperbolic structure of (III.10) in order to define a sub-class of test function ϕ that will allow us to identify the anomalous diffusion limit of this equation. This is the purpose of the following auxiliary problem.

III.4.1 Auxiliary problem

For a smooth function ψ, we define ϕ_ε as the solution of

$$\varepsilon v\cdot\nabla_x\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon = 0 \qquad \forall(t,x,v)\in\mathbb{R}^+\times\Omega\times\mathbb{R}^d, \tag{III.29a}$$

$$\phi^\varepsilon(t,x,0) = \psi(t,x) \qquad \forall(t,x)\in\mathbb{R}^+\times\Omega, \tag{III.29b}$$

$$\gamma_+\phi^\varepsilon(t,x,v) = \gamma_-\phi^\varepsilon\big(t,x,\mathcal{R}_x(v)\big) \qquad \forall(t,x,v)\in\mathbb{R}^+\times\Sigma_+. \tag{III.29c}$$

with $\mathcal{R}_x(v) = v - 2\big(n(x)\cdot v\big)n(x)$ for x in $\partial\Omega$.
Because of the specular reflection boundary condition (III.29c), it is much more challenging to construct a solution ϕ_ε of this problem than it was in the absorption case. In fact, we will see later on that if we want to have enough regularity estimates on ϕ_ε in order to take the limit in the weak formulation of the fractional Vlasov-Fokker-Planck equation, we will need an additional assumption on the initial condition ψ. Setting aside these considerations for the moment, let us show how we can construct ϕ_ε from smooth function ψ through the definition of a function $\eta : \Omega\times\mathbb{R}^d \mapsto \bar{\Omega}$ in the following sense:

Proposition III.1.4. *If Ω is either a half-space or smooth and strongly convex, then there exists a function $\eta : \Omega\times\mathbb{R}^d \to \bar{\Omega}$ such that*

$$\phi^\varepsilon(t,x,v) = \psi\big(t,\eta(x,\varepsilon v)\big) \tag{III.48}$$

is a solution of the auxiliary problem (III.29a)-(III.29b)-(III.29c).

Proof. The proof will consist of two steps. First we construct an appropriate η by identifying the characteristic lines underlying the hyperbolic problem (III.29a)-(III.29c), and then we check that ϕ^ε defined as above is indeed solution of the auxiliary problem.

III.4.1.1 Construction of η

The purpose of η is to follow the characteristic lines defined by (III.29a) and (III.29c). Those lines $(x(s), v(s))$, parametrised by $s \in [0, \infty)$, are given by:

$$\begin{cases} \dot{x}(s) = \varepsilon v(s) & x(0) = x^{in}, \\ \dot{v}(s) = -v(s) & v(0) = v^{in}, \\ \text{If } x(s) \in \partial\Omega \text{ then } v(s^+) = \mathcal{R}_{x(s)}(v(s^-)). \end{cases} \tag{IV.23}$$

Solving this system of ODEs, we see that this trajectory $x(s)$ consists of straight lines with exponentially decreasing velocity $v(s)$ reflected upon hitting the boundary. More precisely, if we denote s_i the times of reflection, i.e. the times for which $x(s_i) \in \partial\Omega$, with the convention $s_0 = 0$, we have for the velocity:

$$\begin{cases} v(s) = e^{-s} v_0 & \text{for } s \in [0, s_1), \\ v(s_i^+) = \mathcal{R}_{x(s_i)} v(s_i^-), \\ v(s) = e^{-(s-s_i)} v(s_i^+) & \text{for } s \in (s_i, s_{i+1}), \end{cases} \tag{III.49}$$

which gives the trajectory, for $s \in (s_i, s_{i+1})$:

$$\begin{aligned} x(s) &= x_0 + \varepsilon \int_0^s v(\tau) \mathrm{d}\tau \\ &= x_0 + \varepsilon \sum_{k=0}^{i-1} \int_{s_k}^{s_{k+1}} v(\tau) \mathrm{d}\tau + \varepsilon \int_{s_i}^s v(\tau) \mathrm{d}\tau \\ &= x_0 + \varepsilon \sum_{k=0}^{i-1} \left(1 - e^{-(s_{k+1} - s_k)}\right) v(s_k^+) + \varepsilon \left(1 - e^{-(s - s_k)}\right) v(s_i^+). \end{aligned}$$

Instead of considering an exponentially decreasing velocity $v(s)$ on an infinite interval $s \in [0, \infty)$, we would like to consider a constant speed on a finite interval $[0, 1)$. To that end, we notice that the reflection operator $\mathcal{R}$ is isometric in the sense that:

$$
\begin{aligned}
v(s_i^+) &= \mathcal{R}_{x(s_i)}\big(v(s_i^-)\big) \\
&= \mathcal{R}_{x(s_i)}\big(e^{-(s_i-s_{i-1})}v(s_{i-1}^+)\big) \\
&= e^{-(s_i-s_{i-1})}\mathcal{R}_{x(s_i)} \circ \mathcal{R}_{x(s_{i-1})}\big(e^{-(s_{i-1}-s_{s-2})}v(s_{i-2}^+)\big) \\
&= e^{-(s_i-s_{i-2})}\mathcal{R}_{x(s_i)} \circ \mathcal{R}_{x(s_{i-1})} \circ \mathcal{R}_{x(s_{i-2})}\big(e^{-(s_{i-2}-s_{s-3})}v(s_{i-3}^+)\big) \\
&= e^{-(s_i-0)}\mathcal{R}_{x(s_i)} \circ \mathcal{R}_{x(s_{i-1})} \circ \cdots \circ \mathcal{R}_{x(s_1)}\big(v_0\big).
\end{aligned}
$$

Furthermore, we introduce the notation R^i denoting:

$$
\begin{cases}
R^0 = Id, \\
R^i = \mathcal{R}_{x(s_i)} \circ R^{i-1},
\end{cases}
\tag{III.50}
$$

and a new velocity $w(s) := e^s v(s)$ which then satisfies:

$$
\begin{cases}
w(s) = v_0 & \text{for } s \in (0, s_1), \\
w(s_i) = R^i v_0, & \\
w(s) = R^i w(s_i) & \text{for } s \in [s_i, s_{i+1}).
\end{cases}
\tag{III.51}
$$

It is easy to check that for any s, $|w(s)| = |v_0|$. The trajectory $x(s)$ can be written, with the velocity $w(s)$ as:

$$
\begin{aligned}
x(s) &= x_0 + \varepsilon \int_0^s e^{-\tau} w(\tau) \mathrm{d}\tau \\
&= x_0 + \varepsilon \sum_{k=0}^{i-1} \left(e^{-s_k} - e^{-s_{k+1}}\right) w(s_k) + \varepsilon \left(e^{-s} - e^{-s_i}\right) w(s_i).
\end{aligned}
$$

Finally, we introduce a new parametrisation $\tau = 1 - e^{-s} \in [0, 1)$ and the corresponding reflection times $\tau_i := 1 - e^{-s_i}$ with which we have, for any $\tau \in [\tau_i, \tau_{i+1})$ with $i \geq 1$:

$$
\begin{cases}
x(\tau) = x_0 + \varepsilon \displaystyle\sum_{k=0}^{i-1} (\tau_{k+1} - \tau_k)\, w(\tau_k) + \varepsilon\, (\tau - \tau_i)\, w(\tau_i), \\
w(\tau) = w(\tau_i) = R^i w_0.
\end{cases}
\tag{III.52}
$$

These trajectories can be seen as geodesic trajectory in a Hamiltonian billiard, as illustrated by Figure III.2. In order to solve (III.29a)-(III.29c) using a characteristic method we would like to define a function η^ε that relates (x_0, w_0) to $x(\tau{=}1)$ (or

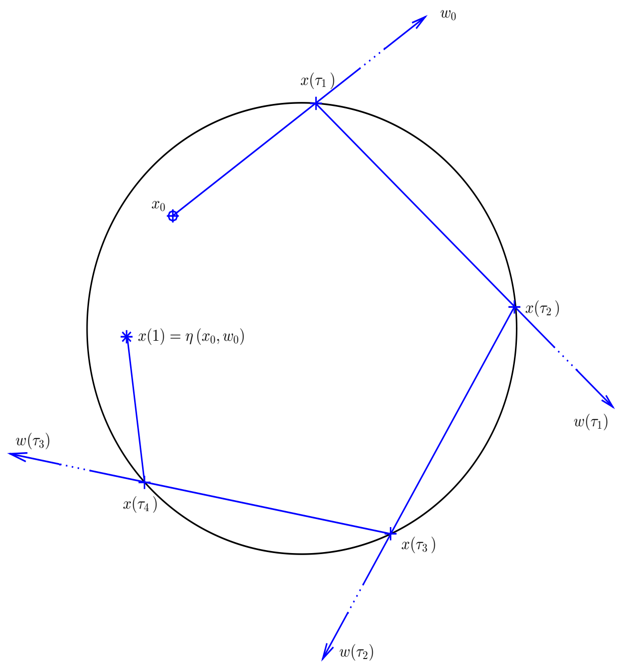


Fig. III.2 Example of trajectory of Ω is a ball of radius 1

$x(s{=}\infty)$ for the initial parametrization). It is natural to construct η^ε by induction on the number of reflections. Such a construction is already well known in the field of mathematical billiards. We refer for instance to the Chapter 2 of the monograph of Chernov-Markarian [CM06] for the construction in dimension 2 and the paper of Halpern [Hal77] where he defines a function $F_t(x, v)$ which gives the position and forward direction of motion of a particle in the billiard, in relation to which our $\eta^\varepsilon(x, v)$ is just the first component of $F_{t=\varepsilon}(x, v)$. To make sure F_t, hence η^ε, is well defined, we just need to make sure that there are no accumulation of reflection times, i.e. that there is only a finite number of reflections occurring during a finite time interval. To that end, we consider the point on the boundary at which these accumulations would happen. Chernov-Markarian explain that it cannot happen on a flat surface and, moreover, in dimension two, Halpern gives a result which can be stated as follows

Theorem. *Let us call ζ the function such that*

$$\Omega = \{x \in \mathbb{R}^d / \zeta(x) < 0\} \text{ and } \partial\Omega = \{x \in \mathbb{R}^d / \zeta(x) = 0\}.$$

If ζ has a bounded third derivative and nowhere vanishing curvature on $\partial\Omega$ in the sense that there exists a constant $C_\zeta > 0$ such that for all $\xi \in \mathbb{R}^d$:

$$\sum_{i,j=1}^{d} \xi_i \frac{\partial^2 \zeta}{\partial x_i \partial x_j} \xi_j \geq C_\zeta |\xi|^2$$

then $F_t(x,v)$ is well defined for all $(x,v) \in \Omega \times \mathbb{R}^d$.

We call *strongly convex* such domains, and this result was later extended by Safarov-Vassilev to higher dimension as stated in Lemma 1.3.17 of [SV97]. We will consider Ω to be a half-space or a ball, neither of which allows for the accumulation of reflection times hence η^ε can be defined as:

$$\eta^\varepsilon(x_0, w_0) = x(\tau{=}1) = x_0 + \varepsilon \sum_{k=0}^{M-1} (\tau_{k+1} - \tau_k)\, w(\tau_k) + \varepsilon\,(1 - \tau_M)\, w(\tau_M) \qquad \text{(III.53)}$$

where $M = M(x_0, w_0)$ is the (finite) number of reflections undergone by the trajectory that starts at (x_0, w_0). Note that this expression yields immediately that for any $(x,v) \in \Omega \times \mathbb{R}^d$:

$$\eta^\varepsilon(x,v) = \eta^1(x, \varepsilon v)$$

so that, from now on, we will forgo the superscript 1 and always consider $\eta(x, \varepsilon v)$.

III.4.1.2 ϕ^ε solution of the auxiliary problem

We now define, for any given smooth function ψ:

$$\phi^\varepsilon(t,x,v) = \psi\big(t, \eta(x, \varepsilon v)\big).$$

By construction, we know that ϕ^ε satisfies (III.29b) and (III.29c). For (III.29a) we differentiate along the characteristic curves:

$$\frac{d}{ds}\phi^\varepsilon(t, x(s), v(s)) = \frac{d}{ds}\psi\Big(t, \eta\big(x(0), \varepsilon v(0)\big)\Big) = 0$$

which yields by (IV.23)

$$\begin{aligned}
\dot{x}(s) \cdot \nabla_x \phi^\varepsilon(x(s), v(s)) + \dot{v}(s) \cdot \nabla_v \phi^\varepsilon(x(s), v(s)) = 0 \\
\varepsilon v(s) \cdot \nabla_x \phi^\varepsilon(x(s), v(s)) - v(s) \cdot \nabla_v \phi^\varepsilon(x(s), v(s)) = 0.
\end{aligned}$$

Take $s = 0$ and you get:

$$\varepsilon v \cdot \nabla_x \phi^\varepsilon(x,v) - v \cdot \nabla_v \phi^\varepsilon(x,v) = 0$$

which concludes the proof of Proposition III.1.4. □

The solution ϕ^ε has a scaling property similar to (III.43) for the solution of the auxiliary problem in the absorption case, namely :

$$\begin{aligned}
\big(-\Delta_v\big)^s\Big[\phi^\varepsilon(t,x,v)\Big] &= c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\psi\big(t,\eta(x,\varepsilon v)\big) - \psi\big(t,\eta(x,\varepsilon w)\big)}{|v-w|^{N+2s}} \,\mathrm{d}w \\
&= \varepsilon^{2s} c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\psi\big(t,\eta(x,\varepsilon v)\big) - \psi\big(t,\eta(x,w)\big)}{|\varepsilon v-w|^{N+2s}} \,\mathrm{d}w \\
&= \varepsilon^{2s}\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)
\end{aligned}$$

Hence, the weak formulation of (III.7a)-(III.7b)-(III.4) becomes:

$$\begin{aligned}
&\iiint\limits_{Q_T} f^\varepsilon \Big(\partial_t \psi - \big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)\Big) \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\
&\qquad + \iint\limits_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\psi\big(0,\eta(x,\varepsilon v)\big)\,\mathrm{d}x\,\mathrm{d}v = 0.
\end{aligned} \tag{III.54}$$

III.4.2 Macroscopic limit

Using the same arguments as in the unbounded or the absorption case, one can show that if $\psi \in \mathcal{D}([0,T)\times\bar{\Omega})$ then

$$\lim_{\varepsilon\searrow 0} \iiint\limits_{Q_T} f^\varepsilon \partial_t \psi\big(t,\eta(x,\varepsilon v\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iint\limits_{(0,T)\times\Omega} \rho(t,x)\psi(t,x)\,\mathrm{d}t\,\mathrm{d}x$$

and

$$\lim_{\varepsilon\searrow 0} \iint\limits_{\Omega\times\mathbb{R}^N} f_{in}(x,v)\phi^\varepsilon(0,x,v)\,\mathrm{d}x\,\mathrm{d}v = \int\limits_{\Omega} \rho_{in}(x)\psi(0,x)\,\mathrm{d}x.$$

For the last term, we prove the following Lemma:

Lemma III.4.1. *If Ω is a half-space or a ball in $\mathbb{R}^d$ then for any $\psi \in \mathfrak{D}_T(\Omega)$ defined as*

$$\mathfrak{D}_T(\Omega) = \Big\{\psi \in \mathcal{C}^\infty([0,T)\times\bar\Omega) \text{ s.t. } \psi(T,\cdot) = 0 \text{ and } \nabla_x\psi(t,x)\cdot n(x) = 0 \text{ on } \partial\Omega\Big\}. \tag{III.28}$$

we have

$$\lim_{\varepsilon\searrow 0} \iiint\limits_{Q_T} f^\varepsilon \big(-\Delta_v\big)^s \Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iint\limits_{(0,T)\times\Omega} \rho(t,x)(-\Delta)^s_{SR}\psi(t,x)\,\mathrm{d}t\,\mathrm{d}x \tag{III.55}$$

where $(-\Delta)^s_{SR}$ is given in Definition III.33 and can equivalently be written as:

$$(-\Delta)^s_{SR}\psi(t,x) = \big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](0). \tag{III.56}$$

Before proving this lemma, which we will do separately for each Ω, let us conclude that with this convergence we can take the limit in (III.54) and see that for all $\psi \in \mathfrak{D}_T(\Omega)$ the macroscopic density $\rho(t,x)$ satisfies

$$\iint\limits_{(0,T)\times\Omega} \rho(t,x)\Big(\partial_t\psi(t,x) - (-\Delta)^s_{\mathrm{SR}}\psi(t,x)\Big)\,\mathrm{d}t\,\mathrm{d}x + \int\limits_{\Omega}\rho_{in}(x)\psi(0,x)\,\mathrm{d}x = 0. \tag{III.57}$$

which ends the proof of Theorem III.1.5.

III.4.2.1 Lemma III.4.1 in a half-space

Consider the half-space $\{x = (x', x_d) \in \mathbb{R}^d : x_d > 0\}$. The function η associated with the half-space can be written explicitly as:

$$\eta(x,v) = \left| \begin{array}{ll} x+v & \text{if } x_d + v_d \geq 0 \\ (x'+v', -x_d - v_d) & \text{if } x_d + v_d \leq 0 \end{array}\right. \tag{III.58}$$

as illustrated by Figure III.3.
We can differentiate $\eta(x,v)$ to see that the Jacobian matrix reads

$$\nabla_v\eta(x,v) = Id + \big(H(x_d+v_d) - 1\big)E_{d,d} \tag{III.59}$$

where $E_{d,d}$ is the matrix with 0 everywhere except the last coefficient (of index d,d) which is 1 and H is the Heaviside function equal to 1 if $x_d + v_d > 0$ and -1 if $x_d + v_d < 0$. Furthermore, the second derivative of $\eta(x,v)$, which we will see as an

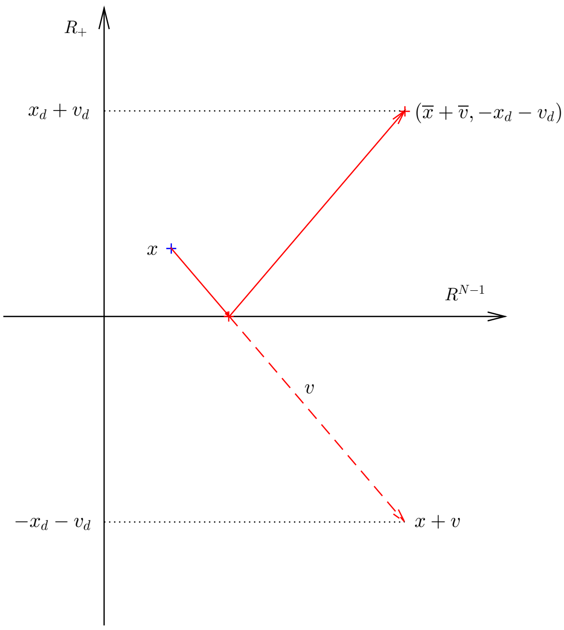


Fig. III.3 Example of trajectory in the half-space

element of $\mathcal{M}_d(\mathbb{R}^d)$, i.e. a vector valued matrix, reads

$$D_v^2\eta(x,v) = 2\big(n \times E_{d,d}\big)\delta_{\eta(x,v)\in\partial\Omega}$$

where n is the outward unit vector of $\partial\Omega$ (which is constant in the half-space), $\delta_{\eta(x,v)\in\partial\Omega}$ is the dirac measure of the boundary surface and $\times$ is a multiplication between a vector $u \in \mathbb{R}^d$ and a matrix $M = (m_{i,j})_{1\le i,j\le d} \in \mathcal{M}_d(\mathbb{R})$ whose result is the vector-valued matrix given by $u \times M = (m_{i,j}u)_{1\le i,j\le d} \in \mathcal{M}_d(\mathbb{R}^d)$.
Furthermore, a straightforward differentiation yields

$$D_v^2\Big[\psi\big(t,\eta(x,v)\big)\Big] = \big(\nabla_v\eta(x,v)\big)^T D^2\psi\big(t,\eta(x,v)\big)\big(\nabla_v\eta(x,v)\big) + D_v^2\eta(x,v)\nabla\psi\big(t,\eta(x,v)\big).$$

where for any $\psi \in \mathfrak{D}_T$ we have

$$D_v^2\eta(x,v)\nabla\psi\big(t,\eta(x,v)\big) = 2\Big(n\cdot\nabla\psi\big(t,\eta(x,v)\big)\Big)E_{d,d}\delta_{\eta(x,v)\in\partial\Omega} = 0$$

since for all $y = \eta(x,v) \in \partial\Omega$ we have $n(y)\cdot\nabla\psi(t,y) = 0$.
To prove Lemma III.4.1 we will show that $\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)$ converges strongly

in $L^\infty(0,T;L^2_{F(v)}(\Omega\times\mathbb{R}^d)$ by a dominated convergence argument. Since f^εconverges weakly in $L^\infty(0,T;L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)$ we can then pass to the limit in the left-hand-side of (III.55) and Lemma III.4.1 follows.
We begin by the proof of point-wise convergence. We introduce the function χ_x : $\mathbb{R}^d\times\mathbb{R}^d\mapsto\mathbb{R}$ given by (omitting the t variable for the sake of clarity)

$$\chi_x(v,w)=\psi\big(\eta(x,v+w)\big)-\psi\big(\eta(x,w)\big). \tag{III.60}$$

For any $(t,x,v)\in Q_T$ we then have

$$\begin{aligned}\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)&-(-\Delta)^s_{\mathrm{SR}}\psi(x)\\&=c_{d,s}P.V.\int_{\mathbb{R}^d}\frac{\psi\big(t,\eta(x,\varepsilon v)\big)-\psi\big(t,\eta(x,\varepsilon v+w)\big)}{|w|^{N+2s}}\,\mathrm{d}w\\&\quad-c_{d,s}P.V.\int_{\mathbb{R}^d}\frac{\psi\big(t,x)-\psi\big(t,\eta(x,w)\big)}{|w|^{N+2s}}\,\mathrm{d}w\\&=c_{d,s}P.V.\int_{\mathbb{R}^d}\frac{\chi_x(\varepsilon v,0)-\chi_x(\varepsilon v,w)}{|w|^{d+2s}}\,\mathrm{d}w.\end{aligned}\tag{III.61}$$

For $\delta>0$, we split the integral as follow

$$\begin{aligned}c_{d,s}P.V.\int_{\mathbb{R}^d}\frac{\chi_x(\varepsilon v,0)-\chi_x(\varepsilon v,w)}{|w|^{d+2s}}\,\mathrm{d}w=c_{d,s}P.V.&\int_{|w|\le\delta}\frac{\chi_x(\varepsilon v,0)-\chi_x(\varepsilon v,w)}{|w|^{d+2s}}\,\mathrm{d}w\\&+c_{d,s}\int_{|w|\ge\delta}\frac{\chi_x(\varepsilon v,0)-\chi_x(\varepsilon v,w)}{|w|^{d+2s}}\,\mathrm{d}w.\end{aligned}$$

On the one hand we see that

$$\begin{aligned}\left|\int_{|w|\ge\delta}\frac{\chi_x(\varepsilon v,0)-\chi_x(\varepsilon v,w)}{|w|^{d+2s}}\,\mathrm{d}w\right|&\le 2\|\chi_x(\varepsilon v,\cdot)\|_{L^\infty(\mathbb{R}^d)}\int_{|w|\ge\delta}\frac{1}{|w|^{d+2s}}\,\mathrm{d}w\\&\le 2\delta^{-2s}\|\chi_x(\varepsilon v,\cdot)\|_{L^\infty(\mathbb{R}^d)}\end{aligned}$$

and by definition of χ_x

$$\sup_w|\chi_x(\varepsilon v,w)|=\sup_w\Big|\psi\big(\eta(x,\varepsilon v+w)\big)-\psi\big(\eta(x,w)\big)\Big|\xrightarrow[\varepsilon\to0]{}0.$$

so the integral over $|w| \geq \delta$ vanishes. On the other hand, using the symmetry of the set $\{|w| \leq \delta\}$ we write

$$\begin{aligned} &P.V. \int_{|w|\leq\delta} \frac{\chi_x(\varepsilon v, 0) - \chi_x(\varepsilon v, w)}{|w|^{N+2s}} \, \mathrm{d}w \\ &\quad = \frac{1}{2} P.V. \int_{|w|\leq\delta} \frac{2\chi_x(\varepsilon v, 0) - \chi_x(\varepsilon v, w) - \chi_x(\varepsilon v, -w)}{|w|^{d+2s}} \, \mathrm{d}w \end{aligned}$$

where we can expand $\chi_x(\varepsilon v, \pm w)$ using a second-order Taylor-Lagrange expansion which yields, for some θ and $\tilde{\theta}$ in the ball $B(\delta)$ centred at the origin with radius δ

$$\begin{aligned} &2\chi_x(\varepsilon v, 0) - \chi_x(\varepsilon v, w) - \chi_x(\varepsilon v, -w) \\ &= -\nabla_w \chi_x(\varepsilon v, 0) \cdot w - w \cdot D^2\chi_x(\varepsilon v, \theta) w - \nabla_w \chi_x(\varepsilon v, 0) \cdot (-w) - (-w) \cdot D^2\chi_x(\varepsilon v, \tilde{\theta})(-w) \\ &= -w \cdot \Big(D^2\chi_x(\varepsilon v, \theta) + D^2\chi_x(\varepsilon v, \tilde{\theta}) \Big) w \end{aligned}$$

therefore

$$\left| P.V. \int_{|w|\leq\delta} \frac{\chi_x(\varepsilon v, 0) - \chi_x(\varepsilon v, w)}{|w|^{N+2s}} \, \mathrm{d}w \right| = \frac{1}{2} \left| \int_{|w|\leq\delta} \frac{w\big(D^2\chi_x(\varepsilon v, \theta) + D^2\chi_x(\varepsilon v, \tilde{\theta})\big) w}{|w|^{d+2s}} \, \mathrm{d}w \right| \tag{III.62}$$

where the P.V. is not needed any more since $s < 1$. For any fixed $\theta \in B(\delta)$, we have

$$\begin{aligned} D^2\chi_x(\varepsilon v, \theta) = &\big(\nabla_v \eta(x, \varepsilon v + \theta)\big)^T D^2\psi\big(\eta(x, \varepsilon v + \theta)\big) \big(\nabla_v \eta(x, \varepsilon v + \theta)\big) \\ &- \big(\nabla_v \eta(x, \theta)\big)^T D^2\psi\big(\eta(x, \theta)\big) \big(\nabla_v \eta(x, \theta)\big). \end{aligned}$$

If $x + \varepsilon v + \theta$ and $x + \theta$ are either both in Ω or both outside Ω then thanks to (III.59) we know that $\nabla_v \eta(x, \varepsilon v + \theta) = \nabla_v \eta(x, \theta)$. We denote M this matrix and we have

$$D^2\chi_x(\varepsilon v, \theta) = M^T \Big(D^2\psi\big(\eta(x, \varepsilon v + \theta)\big) - D^2\psi\big(\eta(x, \theta)\big) \Big) M$$

in which case the regularity of ψ yields

$$\lim_{\varepsilon\to 0} D^2\chi_x(\varepsilon v, \theta) = 0.$$

If x is in the interior of Ω, then for ε and δ small enough, we will obviously have $x + \theta$ and $x + \varepsilon v + \theta$ inside Ω. Moreover, if x is on the boundary $\partial\Omega$ then for any fixed θ in $B(\delta)$, when ε is small enough we will also have $x + \theta$ and $x + \varepsilon v + \theta$ either both inside Ω if $w \cdot n(x) < 0$ or outside Ω if $w \cdot n(x) \geq 0$. As a consequence, we have point-wise convergence of the integrand in the left side of (III.62) therefore (III.59) and the regularity of ψ ensure that we can use dominated convergence in $L^1(B(\delta))$ to write

$$\lim_{\varepsilon\to 0}\left|P.V.\int_{|w|\le\delta}\frac{\chi_x(\varepsilon v,0)-\chi_x(\varepsilon v,w)}{|w|^{N+2s}}\,\mathrm{d}w\right|$$
$$=\frac{1}{2}\left|\int_{|w|\le\delta}\lim_{\varepsilon\to 0}\frac{w\big(D^2\chi_x(\varepsilon v,\theta)+D^2\chi_x(\varepsilon v,\tilde{\theta})\big)w}{|w|^{d+2s}}\,\mathrm{d}w\right|=0.$$

Now that we have proven the point-wise convergence, let us show that

$$v\mapsto\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)$$

is bounded uniformly in ε by a function in $L^2_{F(v)}(\Omega\times\mathbb{R}^d)$. The regularity of ψ and the above computation of the jacobian matrix of η yield in particular that for all $t\in[0,T)$

$$\sup_{v\in\mathbb{R}^d}D^2_v\Big[\psi\big(t,\eta(x,v)\big)\Big]\in L^2(\Omega). \tag{III.63}$$

Therefore, for any $t\in[0,T)$ we introduce $G_t(x)$ given by

$$G_t(x)=\|\psi(t,\cdot)\|_{L^\infty(\Omega)}+\Big\|D^2_v\Big[\psi\big(t,\eta(x,\cdot)\big)\Big]\Big\|_{L^\infty(\mathbb{R}^d)}.$$

As we did before, we can split the integral expression of the fractional Laplacian into a integral on a ball of radius δ around the singularity and an integral on the complement of that ball. For the latter, we write for some constant $C>0$

$$\left|c_{d,s}\int\limits_{\mathbb{R}^d\setminus B(\delta)}\frac{\psi\big(\eta(x,\varepsilon v)\big)-\psi\big(\eta(x,\varepsilon v+w)\big)}{|w|^{d+2s}}\,\mathrm{d}w\right|\le C\|\psi(t,\cdot)\|_{L^\infty(\Omega)}\int\limits_{\mathbb{R}^d\setminus B(\delta)}\frac{1}{|w|^{d+2s}}\,\mathrm{d}w$$
$$\le C\|\psi(t,\cdot)\|_{L^\infty(\Omega)}\delta^{-2s}.$$

For the integral over $B(\delta)$, we use a second order Taylor-Lagrange expansion like we did for χ_x and write

$$\begin{aligned}
&\left| c_{d,s} \int\limits_{B(\delta)} \frac{\psi\big(\eta(x,\varepsilon v)\big) - \psi\big(\eta(x,\varepsilon v + w)\big)}{|w|^{d+2s}} \mathrm{d}w \right| \\
&\qquad \leq C \int\limits_{B(\delta)} \frac{w \cdot \Big(D^2\Big[\psi\big(\eta(x,\cdot)\big)\Big](\varepsilon v + \theta) + D^2\Big[\psi\big(\eta(x,\cdot)\big)\Big](\varepsilon v + \tilde{\theta})\Big) w}{|w|^{d+2s}} \mathrm{d}w \\
&\qquad \leq \Big\| D^2\Big[\psi\big(\eta(x,\cdot)\big)\Big]\Big\|_{L^\infty(\mathbb{R}^d)} \delta^{2-2s}.
\end{aligned}$$

Put together we see that for $\delta = 1$ we have for all $\varepsilon > 0$ and $v \in \mathbb{R}^d$

$$\left| \big(- \Delta_v\big)^s \Big[\psi\big(t, \eta(x,\cdot)\big)\Big](\varepsilon v) \right| \leq G_t(x)$$

and $G_t(x)$ is in $L^2(\Omega)$ by the previous estimates on the second derivative. Hence, we have proven that $\big(- \Delta_v\big)^s \Big[\psi\big(t, \eta(x,\cdot)\big)\Big](\varepsilon v)$ converges strongly in $L^\infty(0,T; L^2_{F(v)}(\Omega \times \mathbb{R}^d))$ to $(-\Delta)^s_{\mathrm{SR}}\psi(t,x)$ and Lemma III.4.1 in the half-space follows.

III.4.2.2 Lemma III.4.1 in a ball

We consider, without loss of generality, that Ω is the unit ball in $\mathbb{R}^d$. For ψ is in $\mathfrak{D}_T(\Omega)$, we will again to prove Lemma III.4.1 by establishing the strong convergence of $\big(- \Delta_v\big)^s \Big[\psi\big(t, \eta(x,\cdot)\big)\Big](\varepsilon v)$ in $L^\infty(0,T; L^2_{F(v)}(\Omega \times \mathbb{R}^d))$ to $(-\Delta)^s_{\mathrm{SR}}\psi(t,x)$.

First, let us point out that the arguments we presented in the half-space to prove the point-wise convergence still hold in the ball. Indeed, we can introduce the function χ_x defined in (III.60) and split (III.61) over $|w| \leq \delta$ and $|w| \geq \delta$ for some $\delta > 0$. On the one hand, if we bound the integral over $|w| \geq \delta$ by the L^∞-norm of χ_x in Ω and the integral of kernel away from its singularity, it follows that this term goes to 0 by definition of χ_x and regularity of ψ. On the other hand, the integral over $|w| \leq \delta$ can be handled exactly the same way as in the half-space. More precisely, if x is away from the boundary then for δ and ε small enough $\eta(x, \varepsilon v + w) = x + \varepsilon v + w$ and there is no issue; and if x is on $\partial\Omega$ then we use the fact that locally the boundary of the ball is isomorphic to the hyperplane $\{x_d = 0\}$ so we recover the previous setting and a dominated convergence argument in $L^1(B(\delta))$ will show that the integral over $|w| \leq \delta$ goes to 0. Together, these two controls and (III.61) prove the point-wise convergence.

The rest of our proof of Lemma III.4.1 requires some estimates on the derivatives of η. These estimates can be established by a detailed analysis of the trajectories described by η and we have devoted the Appendix A of this thesis to this analysis. In particular, in Section A.0.3, we prove the following Lemma:

Lemma III.4.2. *For all $\psi \in \mathfrak{D}_T$ there exists $p > 2$ such that*

$$\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,v)\big)\Big] \in L^p_{F(v)}(\Omega\times\mathbb{R}^d).$$

The strong convergence of $\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,\cdot)\big)\Big](\varepsilon v)$ in $L^2_{F(v)}(\Omega\times\mathbb{R}^d)$ then follows from the following result

Lemma III.4.3. *If $(h_\varepsilon)_{\varepsilon>0}$ converges point-wise to h and is bounded in $L^p_{F(v)}(\Omega\times\mathbb{R}^d)$ for some $p > 2$ uniformly in ε then h_ε converges strongly to h in $L^2_{F(v)}(\Omega\times\mathbb{R}^d)$.*

Proof. Consider $R > 0$ and the ball $B(R)$ of radius R centred at 0 in $\mathbb{R}^d$. The Egorov theorem states that, since $\Omega\times B(R)$ is a bounded domain, for any $\delta > 0$ one can find a subset $A_\delta \subset \Omega\times B(R)$ such that $|\{\Omega\times B(R)\}\setminus A_\delta| \le \delta$ and h_ε converges uniformly on A_δ which means in particular

$$\int_{A_\delta} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v \to 0.$$

As a consequence, we split the norm as follows

$$\begin{aligned}\iint\limits_{\Omega\times\mathbb{R}^d} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v &= \iint\limits_{A_\delta} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v + \iint\limits_{\{\Omega\times B(R)\}\setminus A_\delta} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v \\ &\quad + \iint\limits_{\Omega\times\{\mathbb{R}^d\setminus B(R)\}} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

The first term is handled by Egorov's theorem. For the second, we write

$$\begin{aligned}&\Bigg|\iint\limits_{\{\Omega\times B(R)\}\setminus A_\delta} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v\Bigg| \\ &\le \Bigg(\iint\limits_{\{\Omega\times B(R)\}\setminus A_\delta} |h_\varepsilon - h|^p F(v)\,\mathrm{d}x\,\mathrm{d}v\Bigg)^{2/p}\Bigg(\iint\limits_{\{\Omega\times B(R)\}\setminus A_\delta} F(v)\,\mathrm{d}x\,\mathrm{d}v\Bigg)^{1-2/p} \\ &\le C|\{\Omega\times B(R)\}\setminus A_\delta|^{1-2/p} \\ &\le C\delta^{1-2/p}\end{aligned}$$

and for the third

$$\begin{aligned}
&\Bigg| \iint\limits_{\Omega\times\{\mathbb{R}^d\setminus B(R)\}} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v \Bigg| \\
&\qquad\le \Bigg(\iint\limits_{\Omega\times\{\mathbb{R}^d\setminus B(R)\}} |h_\varepsilon - h|^p F(v)\,\mathrm{d}x\,\mathrm{d}v \Bigg)^{2/p} \Bigg(\iint\limits_{\Omega\times\{\mathbb{R}^d\setminus B(R)\}} F(v)\,\mathrm{d}x\,\mathrm{d}v \Bigg)^{1-2/p} \\
&\qquad\le C\Big(\frac{1}{R^{2s}}\Big)^{1-2/p}.
\end{aligned}$$

Hence, for any $\tilde{\delta} > 0$ we can find R such that $R^{-2s(1-2/p)} \le \tilde{\delta}/3$, δ such that $\delta^{1-2/p} \le \tilde{\delta}/3$ and ε_0 such that for all $\varepsilon \le \varepsilon_0$

$$\int_{A_\delta} |h_\varepsilon - h|^2 F(v)\,\mathrm{d}x\,\mathrm{d}v \le \frac{\tilde{\delta}}{3}$$

and the lemma follows. □

Remark III.4.4. *In both the half-space and the ball, when $s < 1/2$, we do not need to assume that $\nabla\psi(x)\cdot n(x) = 0$ for all x on the boundary which means we can actually extend the set of test functions to $\psi \in \mathcal{C}^\infty([0,T)\times\bar{\Omega})$ with $\psi(T,\cdot) = 0$. Indeed, in those cases, η is regular enough to ensure that $\psi\big(t,\eta(x,v)\big)$ is in $H^1(\mathbb{R}^d)$ with respect to the velocity and since $H^{2s}(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$, the fractional Laplacian of order s of $\psi\big(t,\eta(x,v)\big)$ will be in $L^2_{F(v)}(\Omega\times\mathbb{R}^d)$. Moreover, in our proof of point-wise convergence above, if $2s < 1$ then we can control the singularity for small w in* (III.61) *with a first-order Taylor Lagrange expansion which mean we do not require any assumption on $\nabla\psi$ at the boundary.*

III.5 Well posedness of the specular diffusion equation

This last section is devoted to the proof of Theorem III.1.6 and is divided in three steps. First, we establish some properties of the specular diffusion operator $(-\Delta)^s_{\mathrm{SR}}$. Secondly, we handle the first part of Theorem III.1.6 which is the existence and uniqueness of a weak solution to the specular diffusion equation (III.35a)-(III.35b). Thirdly, we will show that the distributional solution ρ that we constructed in the previous section is precisely this unique weak solution when Ω is either the half-space $\mathbb{R}^d_+ = \{(\bar{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$ or the unit ball B_1 in $\mathbb{R}^d$.

Note that although the theorem holds in both domains and the steps are similar in both cases, the techniques we use at each step often differ so we will have to treat the cases separately several times.

III.5.1 Properties and estimates of the specular diffusion operator

III.5.1.1 $(-\Delta)^s_{\mathrm{SR}}$ on the half-space

When Ω is the half-space $\mathbb{R}^d_+$, $(-\Delta)^s_{\mathrm{SR}}$ can be written as a kernel operator

Proposition III.5.1. *Let us define $K_{\mathbb{R}^d_+}$ as*

$$K_{\mathbb{R}^d_+}(x,y) = c_{d,s}\left(\frac{1}{|x-y|^{d+2s}} + \frac{1}{|(\bar{x}-\bar{y}, x_d+y_d)|^{d+2s}}\right) \tag{III.64}$$

Then we have

$$(-\Delta)^s_{SR}\psi(x) = P.V.\int_{\mathbb{R}^d_+} \big(\psi(x)-\psi(y)\big) K_{\mathbb{R}^d_+}(x,y)\,\mathrm{d}y. \tag{III.34}$$

Moreover, this kernel is symmetric: $K_{\mathbb{R}^d_+}(x,y) = K_{\mathbb{R}^d_+}(y,x)$ for all x and y in $\mathbb{R}^d_+$ and satisfies

$$c_{d,s}\frac{1}{|x-y|^{d+2s}} \leq K_{\mathbb{R}^d_+}(x,y) \leq c_{d,s}\frac{2}{|x-y|^{d+2s}} \tag{III.65}$$

Proof. The expression for $\det \nabla_v \eta(x,v)$ in the half-space is given by (III.58). Defined as such, $K_{\mathbb{R}^d_+}$ is obviously well defined, although singular, and moreover we have:

$$\begin{aligned} K_{\mathbb{R}^d_+}(x,y) &= c_{d,s}\left(\frac{1}{|x-y|^{d+2s}} + \frac{1}{|(\bar{x}-\bar{y}, x_d+y_d)|^{d+2s}}\right) \\ &= c_{d,s}\left(\frac{1}{|y-x|^{d+2s}} + \frac{1}{|(\bar{y}-\bar{x}, y_d+x_d)|^{d+2s}}\right) = K_{\mathbb{R}^d_+}(y,x). \end{aligned}$$

Finally, since $1/|(\bar{y}-\bar{x}, y_d+x_d)|^{d+2s} \geq 0$, the left-hand-side of (III.65) holds and, as can be seen in Figure III.3, $|(\bar{x}-\bar{y}, x_d+y_d)| \geq |x-y|$ which yields the right-hand-side of (III.65). $\square$

In more general domains Ω, we can also try to write $(-\Delta)^s_{\mathrm{SR}}$ as a kernel operator. The general form of this kernel is given by a generalized change of variable formula, c.f.

[LM95] and reads

$$K_\Omega(x,y) = c_{d,s} \sum_{v \in \eta_x^{-1}(y)} \frac{\big|\det \nabla_v \eta(x,v)\big|^{-1}}{|v|^{d+2s}}. \tag{III.66}$$

where $\eta_x^{-1}(y) = \{v \in \mathbb{R}^d : \eta(x,v) = y\}$. For instance, when Ω is a stripe and a cube, one can show that the Jacobian determinant of η in those domains is bounded away from 0, that the sum is infinite but countable and as a consequence that the kernel will be well defined, symmetric and its singularity will be comparable with the singularity of $\big(-\Delta\big)^s$ as expressed in (III.65) for the half-space. Although we won't dwell on those domains in this paper, we will make sure not to use the explicit expression of the kernel in the half-space as long as we can in order to establish results that will also hold in any domains where the kernel is well defined, symmetric and $2s$-singular. In particular, we can establish an integration by parts formula for $(-\Delta)^s_{\mathrm{SR}}$ from which we will deduce its symmetry.

Proposition III.5.2. *The operator $(-\Delta)^s_{SR}$ satisfies an integration by parts formula: for any ψ and ϕ smooth enough:*

$$\int_\Omega \phi(x)(-\Delta)^s_{SR}\psi(x)\,\mathrm{d}x = \frac{1}{2}\iint_{\Omega\times\Omega} \big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{III.67}$$

Proof. First, we use the kernel operator expression (III.34) for the $(-\Delta)^s_{\mathrm{SR}}$ operator and inverse the variables x and y, using the symmetry of the kernel K_Ω, in order to write the following:

$$\begin{aligned}\int_\Omega \phi(x)(-\Delta)^s_{\mathrm{SR}}\psi(x)\,\mathrm{d}x = &\frac{1}{2}\int_{x\in\Omega}\phi(x)P.V.\int_{y\in\Omega}\big(\psi(x)-\psi(y)\big)K_\Omega(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &-\frac{1}{2}\int_{y\in\Omega}\phi(y)P.V.\int_{x\in\Omega}\big(\psi(x)-\psi(y)\big)K_\Omega(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

In first integral, we add and subtract $(x-y)\nabla\psi(x)\mathbb{1}_{B(x)}(y)$ where $\mathbb{1}_{B(x)}(y)$ is the indicator function of a ball around x included in Ω, and we notice that since ψ is smooth it satisfies for any $x\in\Omega$ and $y\in B(x)$:

$$\psi(x)-\psi(y)-(x-y)\nabla\psi(x)\mathbb{1}_{B(x)}(y) = O\big(|x-y|^2\big)$$

so that the integral

$$\iint\limits_{\Omega\times\Omega} \phi(x)\Big(\psi(x)-\psi(y)-(x-y)\nabla\psi(x)\mathbb{1}_{B(x)}(y)\Big)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

is well defined without need of a principal value because the kernel is $2s-$singular with $2s<2$. We do the same in the second integral, adding and subtracting $(x-y)\nabla\psi(y)\mathbb{1}_{B(y)}(x)$ where $\mathbb{1}_{B(y)}(x)$ is the indicator function of a ball around y included in Ω so that we get:

$$\begin{aligned}\int\limits_\Omega \phi(x)(-\Delta)^s_{\mathrm{SR}}\psi(x)\,\mathrm{d}x &= \frac{1}{2}\iint\limits_{\Omega\times\Omega}\phi(x)\Big(\psi(x)-\psi(y)-(x-y)\nabla\psi(x)\mathbb{1}_{B(x)}(y)\Big)K_\Omega\,\mathrm{d}x\,\mathrm{d}y\\ &+\frac{1}{2}\int\limits_{x\in\Omega}\phi(x)\nabla\psi(x)P.V.\int\limits_{y\in\Omega}(x-y)\mathbb{1}_{B(x)}(y)K_\Omega(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ &-\frac{1}{2}\iint\limits_{\Omega\times\Omega}\phi(y)\Big(\psi(x)-\psi(y)-(x-y)\nabla\psi(y)\mathbb{1}_{B(y)}(x)\Big)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &-\frac{1}{2}\int\limits_{y\in\Omega}\phi(y)\nabla\psi(y)P.V.\int\limits_{x\in\Omega}(x-y)\mathbb{1}_{B(y)}(x)K_\Omega(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Since we can use Fubini's theorem in the first and the third term, we sum both of them and notice that $\big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)=O\big(|x-y|^2\big)$ in order to write

$$
\begin{aligned}
&\frac{1}{2}\iint\limits_{\Omega\times\Omega}\phi(x)\Big(\psi(x)-\psi(y)-(x-y)\nabla\psi(x)\mathbb{1}_{B(x)}(y)\Big)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad-\frac{1}{2}\iint\limits_{\Omega\times\Omega}\phi(y)\Big(\psi(x)-\psi(y)-(x-y)\nabla\psi(y)\mathbb{1}_{B(y)}(x)\Big)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{2}\iint\limits_{\Omega\times\Omega}\Big[\big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)-\phi(x)\nabla\psi(x)\mathbb{1}_{B(x)}(y)(x-y)\\
&\qquad+\phi(y)\nabla\psi(y)\mathbb{1}_{B(y)}(x)(x-y)\Big]K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{2}\iint\limits_{\Omega\times\Omega}\big(\phi(x)-\phi(y)\big)\big(\psi(x)-\psi(y)\big)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad-\frac{1}{2}\int\limits_{x\in\Omega}\phi(x)\nabla\psi(x)P.V.\int\limits_{y\in\Omega}(x-y)K_\Omega(x,y)\mathbb{1}_{B(x)}(y)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad+\frac{1}{2}\int\limits_{y\in\Omega}\phi(y)\nabla\psi(y)P.V.\int\limits_{x\in\Omega}(x-y)\mathbb{1}_{B(y)}(x)K_\Omega(x,y)\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

which concludes the proof. □

As a direct corollary of this proof, we see that since the kernel K_Ω is symmetric, the operator is symmetric as well:

$$
\int\limits_\Omega\phi(x)(-\Delta)^s_{\mathrm{SR}}\psi(x)\,\mathrm{d}x=\int\limits_\Omega\psi(x)(-\Delta)^s_{\mathrm{SR}}\phi(x)\,\mathrm{d}x.
$$

III.5.1.2 $(-\Delta)^s_{\mathrm{SR}}$ on a ball

In the ball, if we wanted to write $(-\Delta)^s_{\mathrm{SR}}$ as a kernel operator using (III.66), the kernel would only be defined almost everywhere because the determinant of $\nabla_v\eta$ is not bounded away from 0. Indeed, as can be seen in Appendix A, for a fixed x, a fixed direction $\theta=v/|v|\in\mathbb{S}^{d-1}$ and a fixed number of reflections, we can find one and only one norm $|v|$ such that the determinant of $\nabla_x\eta(x,|v|\theta)$ is null. This can be seen in the expression (A.9) because finding this norm is equivalent to solving $\det\nabla_v\eta(x,v)=0$ after fixing all the variables except l_{end} and, in that setting, the Jacobian determinant is a monotonous function of l_{end} that passes through 0. However, for each fixed x, the set of velocities v such that the determinant is null is a countable sum of curves since

for each fixed number of reflections k there is exactly one v in that set per direction θ in $\mathbb{S}^{d-1}$. Therefore, the kernel is defined almost everywhere.
Nevertheless, even if we can't rigorously write it with a kernel, the specular diffusion operator still has interesting properties, as for instance:

Proposition III.5.3. *When Ω is a ball B, the operator $(-\Delta)^s_{SR}$ admits the following integration by parts formula: for all ϕ and ψ smooth enough*

$$\int_\Omega \phi(x)(-\Delta)^s_{SR}\psi(x)\,\mathrm{d}x = \frac{1}{2}c_{d,s}\iint_{\Omega\times\mathbb{R}^d}\Big(\phi(x)-\phi\big(\eta(x,v)\big)\Big)\Big(\psi(x)-\psi\big(\eta(x,v)\big)\Big)\frac{\mathrm{d}v\,\mathrm{d}x}{|v|^{d+2s}}. \tag{III.68}$$

From which we readily deduce its symmetry

$$\int_\Omega \phi(x)(-\Delta)^s_{\mathrm{SR}}\psi(x)\,\mathrm{d}x = \int_\Omega \psi(x)(-\Delta)^s_{\mathrm{SR}}\phi(x)\,\mathrm{d}x \tag{III.69}$$

Proof. We write

$$\begin{aligned}\int_\Omega \phi(x)(-\Delta)^s_{\mathrm{SR}}\psi(x)\,\mathrm{d}x &= c_{d,s}\iint_{\Omega\times\mathbb{R}^d}\Big(\phi(x)-\phi\big(\eta(x,v)\big)\Big)\Big(\psi(x)-\psi\big(\eta(x,v)\big)\Big)\frac{\mathrm{d}v\,\mathrm{d}x}{|v|^{d+2s}}\\ &\quad - c_{d,s}P.V.\iint_{\Omega\times\complement\mathbb{R}^d}\phi\big(\eta(x,v)\big)\Big(\psi(x)-\psi\big(\eta(x,v)\big)\Big)\frac{\mathrm{d}v\,\mathrm{d}x}{|v|^{d+2s}}.\end{aligned}$$

In the second term on the right-hand-side we want to do a change of variable $F(x,v) = (y,w)$ such that the trajectory described by η from (y,w) is exactly the trajectory from (x,v) backwards. In particular, that means $\eta(y,w) = x$ and $\eta(x,v) = y$. We have the following result on this change of variable which will be proven in Section A.0.4 of the appendices:

Lemma III.5.4. *The change for variable F given by*

$$F\begin{pmatrix} x \\ v\end{pmatrix} = \begin{pmatrix}\eta(x,v)\\ -\big[\nabla_v\eta(x,v)\big]v\end{pmatrix} \tag{III.70}$$

is precisely the change of variable such that $\eta(F(x,v)) = x$ and the trajectory described by η starting at $\eta(x,v)$ with velocity $-\big[\nabla_v\eta(x,v)\big]v$ is exactly the trajectory from (x,v)

backwards. Moreover, for all (x,v)*:*

$$\det \nabla F(x,v) = 1. \tag{III.71}$$

The singularity that requires the principal value is at $\{v=0\}$ around which we have explicitly $\eta(x,v) = x+v$ hence it will become, through the change of variable, a singularity at $\{w=0\}$ since we have $w=-v$ in the neighbourhood of 0. The change of variables yields

$$\begin{aligned}\int_\Omega \phi(x)(-\Delta)^s_{\mathrm{SR}}\psi(x)\,\mathrm{d}x = c_{d,s} \iint_{\Omega\times\mathbb{R}^d} \Big(\phi(x) - \phi\big(\eta(x,v)\big)\Big)\Big(\psi(x) - \psi(\eta(x,v))\Big)\frac{\mathrm{d}v\,\mathrm{d}x}{|v|^{d+2s}} \\ - c_{d,s} P.V. \iint_{\Omega\times\mathbb{R}^d} \phi(y)\Big(\psi\big(\eta(y,w)\big) - \psi(y)\Big)\frac{\mathrm{d}w\,\mathrm{d}y}{|w|^{d+2s}}\end{aligned}$$

and the integration by parts formula follows. □

Finally, in relation with (III.65), one can see immediately from looking at the integration by part formula in a ball, that the singularity in the operator is of order exactly $2s$.

III.5.1.3 The Hilbert space $\mathcal{H}^s_{\mathbf{SR}}(\Omega)$

We conclude the analysis of $(-\Delta)^s_{\mathrm{SR}}$ by introducing the associated Hilbert space $\mathcal{H}^s_{\mathrm{SR}}(\Omega)$. This comes down to interpreting the integration by parts formula as a type of scalar product and considering the associated semi-norm in the spirit of the Gagliardo (semi-)norm on the fractional Sobolev space $H^s(\mathbb{R}^d)$ and its relation with the fractional Laplacian as presented e.g. in [DPV12]. The natural semi-norm associated with the specular diffusion operator reads in the half-space

$$[\psi]^2_{\mathcal{H}^s_{\mathrm{SR}}(\mathbb{R}^d_+)} = \frac{1}{2}\iint_{\mathbb{R}^d_+\times\mathbb{R}^d_+} \big(\psi(x) - \psi(y)\big)^2 K_{\mathbb{R}^d_+}(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

and in the ball

$$[\psi]^2_{\mathcal{H}^s_{\mathrm{SR}}(B)} = \frac{c_{d,s}}{2}\iint_{\mathbb{R}^d\times B} \Big(\psi(x) - \psi\big(\eta(x,v)\big)\Big)^2 \frac{1}{|v|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}v.$$

Consequently, we introduce a Hilbert space associated with the specular diffusion operator.

Definition III.5.1. *We define the Hilbert space $\mathcal{H}^s_{SR}(\Omega)$ as*

$$\mathcal{H}^s_{SR}(\Omega) = \Big\{\psi \in L^2(\Omega) : [\psi]_{\mathcal{H}^s_{SR}(\Omega)} < \infty\Big\} \tag{III.72}$$

associated with a scalar product which, on a half-space, read

$$\langle\psi|\phi\rangle_{\mathcal{H}^s_{SR}(\mathbb{R}^d_+)} = \int\limits_{\mathbb{R}^d_+} \psi\phi \,\mathrm{d}x + \frac{1}{2}\iint\limits_{\mathbb{R}^d_+\times\mathbb{R}^d_+} \big(\phi(t,x)-\phi(t,y)\big)\big(\psi(t,x)-\psi(t,y)\big)K_{\mathbb{R}^d_+}(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{III.73}$$

and on the ball becomes

$$\begin{aligned}\langle\psi|\phi\rangle_{\mathcal{H}^s_{SR}(B)} = &\int\limits_B \psi\phi\,\mathrm{d}x\\ &+\frac{c_{d,s}}{2}\iint\limits_{\mathbb{R}^d\times B}\Big(\phi(t,x)-\phi\big(t,\eta(x,v)\big)\Big)\Big(\psi(t,x)-\psi\big(t,\eta(x,v)\big)\Big)\frac{\mathrm{d}x\,\mathrm{d}v}{|v|^{d+2s}}\end{aligned} \tag{III.74}$$

hence the norm associated with $\mathcal{H}^s_{SR}(\Omega)$ is naturally

$$\|\psi\|^2_{\mathcal{H}^s_{SR}(\Omega)} = \|\psi\|^2_{L^2(\Omega)} + [\psi]^2_{\mathcal{H}^s_{SR}(\Omega)}$$

This functional space is strongly linked with the Sobolev space $H^s(\Omega)$ and we refer the interested reader to [DPV12] for more details. We notice right away that $(-\Delta)^s_{\mathrm{SR}}$ is self-adjoint on the Hilbert space $\mathcal{H}^s_{\mathrm{SR}}(\Omega)$ and also, by the estimates on the singularity of the operator established above, we see that $\mathcal{H}^s_{\mathrm{SR}}(\Omega) \subset H^s(\Omega)$.

III.5.2 Existence and uniqueness of a weak solution for the macroscopic equation

We now turn to the specular diffusion equation (III.35a)-(III.35b).

Theorem III.1.6 (Part I)**.** *Let Ω be a half-space or a ball in $\mathbb{R}^d$, u_{in} be in $L^2((0,T)\times\Omega)$ and s be in $(0,1)$. For any $T>0$, there exists a unique weak solution $u \in L^2(0,T;\mathcal{H}^s_{SR}(\Omega))$ of*

$$\partial_t u + (-\Delta)^s_{SR} u = 0 \qquad (t,x) \in [0,T)\times\Omega \tag{III.35a}$$

$$u(0,x) = u_{in}(x) \qquad x\in\Omega \tag{III.35b}$$

in the sense that for any $\psi \in \mathfrak{D}_T$ *defined in* (III.28)*,* u *satisfies if* Ω *is a half-space:*

$$\begin{aligned}&\iint\limits_{(0,T)\times\Omega} u\partial_t\psi \,\mathrm{d}t\,\mathrm{d}x - \int\limits_{\Omega} u_{in}(x)\psi(0,x)\,\mathrm{d}x\\&-\frac{1}{2}\iiint\limits_{(0,T)\times\Omega\times\Omega}\big(u(t,x)-u(t,y)\big)\big(\psi(t,x)-\psi(t,y)\big)K(x,y)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0.\end{aligned}\tag{III.36}$$

and if Ω *is the unit ball*

$$\begin{aligned}&\iint\limits_{(0,T)\times\Omega} u\partial_t\psi \,\mathrm{d}t\,\mathrm{d}x - \int\limits_{\Omega} u_{in}(x)\psi(0,x)\,\mathrm{d}x\\&-\frac{1}{2}\iiint\limits_{(0,T)\times\Omega\times\mathbb{R}^d}\big(u(t,x)-u\big(t,\eta(x,v)\big)\big)\big(\psi(t,x)-\psi\big(t,\eta(x,v)\big)\big)\frac{\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v}{|v|^{d+2s}} = 0.\end{aligned}\tag{III.37}$$

Proof of Theorem III.1.6, (Part I). This proof is strongly inspired by the proof of existence and uniqueness of weak solutions to the Vlasov-Poisson-Fokker-Planck equation from Carrillo [Car98]. We consider an associated problem which comes formally from deriving (III.35a) for $\bar u(t,x) = e^{-\lambda t}u(t,x)$ for some $\lambda > 0$:

$$\begin{aligned}&\partial_t\bar u(t,x) + \lambda\bar u(t,x) + (-\Delta)^s_{\mathrm{SR}}\bar u(t,x) = 0 \quad (t,x)\in(0,T)\times\Omega\\&\bar u(0,x) = \bar u_{in}(x) \qquad\qquad\qquad\qquad\qquad\qquad x\in\Omega.\end{aligned}\tag{III.76}$$

Note that we do not prescribe any explicit boundary condition on $\partial\Omega$. A weak solution of (III.76) is a function $\bar u \in L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega))$ such that for any $\psi\in\mathfrak{D}_T$,

$$\iint\limits_{(0,T)\times\Omega}\Big(-\bar u\partial_t\psi + \lambda\bar u\psi + \bar u(-\Delta)^s_{\mathrm{SR}}\psi\Big)\,\mathrm{d}t\,\mathrm{d}x + \int\limits_{\Omega}\bar u_{in}(x)\psi(0,x)\,\mathrm{d}x = 0.$$

We first prove existence of weak solutions of this problem using a Lax-Milgram argument and we will show afterwards that it implies existence for (III.35a)-(III.35b). We consider on $\mathfrak{D}_T$ the prehilbertian norm

$$|\psi|^2_{\mathfrak{D}_T} = \|\psi\|^2_{\mathcal{H}^s_{\mathrm{SR}}(\Omega)} + \frac{1}{2}\|\psi(0,\cdot)\|^2_{L^2(\Omega)}.$$

We then introduce the bilinear form a from $L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega)) \times \mathfrak{D}_T$ to $\mathbb{R}$ defined as

$$a(\bar{u},\psi) = \iint\limits_{(0,T)\times\Omega} \Big(-\bar{u}\partial_t\psi + \lambda\bar{u}\psi + \bar{u}(-\Delta)^s_{\mathrm{SR}}\psi \Big)\,\mathrm{d}t\,\mathrm{d}x$$

and the continuous bounded linear operator L on F:

$$L(\psi) = \int\limits_{\Omega} \bar{u}_{in}(x)\psi(0,x)\,\mathrm{d}x.$$

From Lemma III.4.2 we know in particular that $\mathfrak{D}_T$ is a subset of $L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega))$ with a continuous injection. Moreover, it is easy to see that a is continuous and it is also coercive since:

$$a(\psi,\psi) = \iint\limits_{(0,T)\times\mathbb{R}^d} \lambda\psi^2 + \psi(-\Delta)^s_{\mathrm{SR}}\psi\,\mathrm{d}t\,\mathrm{d}x + \frac{1}{2}\int\limits_{\Omega}\psi(0,x)^2\,\mathrm{d}x \geq \min(1,\lambda)|\psi|^2_{\mathfrak{D}_T}$$

hence, the Lax-Milgram theorem gives us existence of a weak solution of (III.76) in $L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega))$. From this weak solution ψ we define $\bar{\psi}(t,x) = e^{-\lambda t}\psi(t,x)$ which is obviously in $L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega))$ and weak solution of (III.35a)-(III.35b). Since the equation is linear, to show uniqueness is equivalent to proving that the only weak solution with initial data $u_{in} = 0$ is the zero function. Call u_0 this weak solution. Multiplying (III.35a) by u_0 and integrating over Ω we have:

$$\int\limits_{\Omega}\frac{1}{2}\partial_t\big(u_0^2\big)\,\mathrm{d}x = -\int\limits_{\Omega} u_0(-\Delta)^s_{\mathrm{SR}}u_0\,\mathrm{d}x \leq 0.$$

Hence $\|u_0(t,\cdot)\|_{L^2(\Omega)}$ is decreasing. Since it was 0 to start with, that means $u_0 \equiv 0$ and that concludes the proof of uniqueness of solution. Finally, we notice that the integration by parts formula (III.67) concludes the proof existence and uniqueness of a weak solution of (III.35a)-(III.35b) in the sense given in Theorem III.1.6.

□

III.5.3 Identifying the macroscopic density as the unique weak solution

Finally, we turn to the last part of Theorem III.1.6

Theorem III.1.6 (Part II). *If Ω is a ball or a half-space, the macroscopic density ρ who satisfies* (III.32) *for all $\psi \in \mathfrak{D}_T(\Omega)$ is the unique weak solution of* (III.35a)-(III.35b).

Proof. In order to prove this theorem we will show that there is a unique distributional solution of (III.32), i.e. a unique ρ such that (III.32) holds for all $\psi \in \mathfrak{D}_T(\Omega)$. Indeed, since it is obvious that the weak solution of (III.35a)-(III.35b) is also a distributional solution of (III.32), if we prove its uniqueness then Theorem III.1.6 Part II will follow immediately.
As usual, to prove uniqueness for linear PDEs, we assume that there are two distributional solutions ρ_1 and ρ_2 or (III.32) and we consider their difference $\bar{\rho} = \rho_1 - \rho_2$ which satisfies for any ψ in $\mathfrak{D}_T$

$$\iint\limits_{[0,T)\times\Omega} \bar{\rho}\Big(\partial_t\psi - (-\Delta)^s_{\mathrm{SR}}\psi\Big)\,\mathrm{d}t\,\mathrm{d}x = 0 \tag{III.77}$$

with $\int_\Omega \bar{\rho}\,\mathrm{d}x = 0$ thanks to the conservation of mass. We want to prove that $\bar{\rho}$ is null. In order to do so, we first introduce the following reverse evolution problem and show its wellposedness:

Proposition III.5.5. *For any $\bar{\rho} \in L^\infty([0,T); L^2(\Omega))$ there exists a unique $\psi_{\bar{\rho}}$ weak solution in $L^2((0,T)\times\Omega)$ of:*

$$\begin{cases} \partial_t\psi_{\bar{\rho}} - (-\Delta)^s_{SR}\psi_{\bar{\rho}} = \bar{\rho} & (t,x) \in [0,T)\times\Omega \\ \psi_{\bar{\rho}}(T,x) = 0 & x \in \Omega \end{cases} \tag{III.78}$$

Proof. The proof of part 1 of Theorem III.1.6 above can easily be adapted to show existence of uniqueness of weak solution in $L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega))$ of (III.35a)-(III.35b) with a source term S, namely:

$$\begin{aligned} &\partial_t u + (-\Delta)^s_{\mathrm{SR}} u = S(t,x) && (t,x) \in [0,T)\times\Omega \\ &u(0,x) = u_{in}(x) && x \in \Omega. \end{aligned}$$

To do so, one only needs to change the continuous bounded linear map L to

$$L(\psi) = \int\limits_\Omega \bar{u}_{in}(x)\psi(0,x)\,\mathrm{d}x + \iint\limits_{(0,T)\times\Omega} \bar{S}\psi\,\mathrm{d}t\,\mathrm{d}x$$

where $\bar{S}(t,x) = e^{-\lambda t} S(t,x)$, and the rest of the proof holds. Hence, if we consider this weak solution u and define $\psi_{\bar\rho}(t,x) = u(T-t,x)$ as well as choose S such that $\bar\rho(t,x) = -S(T-t,x)$ and take $u_{in} = 0$, this gives us the unique $\psi_{\bar\rho}$ weak solution of (III.78) in $L^2(0,T;\mathcal{H}^s_{\mathrm{SR}}(\Omega))$. $\square$

We see now that if we can use $\psi_{\bar\rho}$ as a test function in (III.77) then we will have

$$\iint\limits_{[0,T)\times\Omega} \bar\rho^2 \,\mathrm{d}x\,\mathrm{d}t = 0$$

which concludes the proof of uniqueness of the distributional solution ρ of (III.57). It remains to show that $\psi_{\bar\rho}$ is an admissible test function for (III.32).
When Ω is a ball or a half-space and $s < 1/2$, as stated in Remark III.4.4, we don't need to control the second derivative of $\psi(t,\eta(x,v))$ in order to take the limit in the weak formulation (III.54) so we can actually extend the set of test functions to $\mathcal{C}^\infty([0,T)\times\bar\Omega)$ which is dense in $L^\infty([0,T);\mathcal{H}^s_{\mathrm{SR}}(\Omega))$ with respect to the $\mathcal{H}^s_{\mathrm{SR}}$-norm and the result is immediate.
When $s > 1/2$, however, the test functions in (III.77) need to be in $\mathfrak{D}_T$ so we need to understand the behaviour of $\psi_{\bar\rho}$ on the boundary. Let us recall that $\mathfrak{D}_T$ is defined as:

$$\mathfrak{D}_T(\Omega) = \Big\{\psi \in \mathcal{C}^\infty([0,T)\times\bar\Omega) \text{ s.t. } \psi(T,\cdot) = 0 \text{ and } \forall x \in \partial\Omega : \nabla_x\psi(t,x)\cdot n(x) = 0\Big\}.$$

The interaction between the singularity in the specular diffusion operator and the boundary leads us to believe that $\psi_{\bar\rho}$ satisfies a rather strong, non-local boundary condition but we are unable to write this condition explicitly since it is contained in the action of $(-\Delta)^s_{\mathrm{SR}}$. As a consequence, we will show instead that $\psi_{\bar\rho}$ satisfies, in particular, an homogeneous Neumann condition. To that end, we first regularize with respect to time the right hand side of (III.78), and call n the regularizing parameter. For each n, since the operator $(-\Delta)^s_{\mathrm{SR}}$ is self-adjoint and dissipative it generates a strongly continuous semi-group of contractions and as a consequence one can prove, see [Paz83] Section 4.2 for more details, that there exists a unique strong solution ψ_n of (III.78) which, in particular, satisfies for any t

$$(-\Delta)^s_{\mathrm{SR}}\psi_n(t,x) \in L^\infty(\Omega). \tag{III.79}$$

Moreover, we have the following lemma:

Lemma III.5.6. *Let Ω be a ball or a half-space and $s > 1/2$. For any ψ such that $(-\Delta)^s_{SR}\psi(x) \in L^\infty(\Omega)$, we have*

$$\nabla_x \psi(t,x) \cdot n(x) = 0 \qquad \forall x \in \partial\Omega. \tag{III.80}$$

Postponing the proof of this lemma, let us conclude the proof of Theorem III.1.6. For each ψ_n, since it satisfies (III.80), we know that if it was in $H^2(\Omega)$ then we could approach it by functions in $\mathfrak{D}_T$ with respect to the H^2-norm. Moreover, ψ_n is a strong solution of (III.78) which entails that it belongs at least to $\mathcal{H}^{2s}_{\mathrm{SR}}(\Omega)$ because $(-\Delta)^s_{\mathrm{SR}}\psi_n$ is in $L^2(\Omega)$. Further, as stated before, $\mathcal{H}^{2s}_{\mathrm{SR}}(\Omega) \subset H^{2s}(\Omega)$, and since $s < 1$ that means we can approach ψ_n by functions in $\mathfrak{D}_T$ with respect to the $\mathcal{H}^{2s}_{\mathrm{SR}}(\Omega)$-norm, which is strong enough to take the limit in (III.77). Hence, $\psi_{\bar{\rho}}$ is an admissible test function for (III.77), which yields the uniqueness of the distributional solution of (III.57). □

Proof of Lemma III.5.6. For the half-space, we notice that $(-\Delta)^s_{\mathrm{SR}}\psi$ can be interpreted as the fractional Laplacian acting on its mirror-extension $\widetilde{\psi}$ defined as:

$$\widetilde{\psi}(t,x) = \left| \begin{array}{ll} \psi(t,x) & \text{if } x_d \geq 0 \\ \psi(t,[x',-x_d]) & \text{if } x_d \leq 0 \end{array} \right. \tag{III.81}$$

where we wrote $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$. The boundary behaviours of ψ follows readily because we know that in order for $\big(-\Delta\big)^s\widetilde{\psi}$ to be bounded, $\widetilde{\psi}$ has to be at least $\mathcal{C}^{1,2s-1}$ on $\mathbb{R}^d$. Since it is a mirror-extension that means ψ has to satisfy an homogeneous Neumann condition on the boundary:

$$\nabla_x \psi(t,x) \cdot n(x) = 0 \qquad \forall x \in \partial\Omega.$$

Note that the same line of argument would also hold in a stripe or a cube since we can define in those cases an extension that consists of a composition of mirror extensions and such that $(-\Delta)^s_{\mathrm{SR}}\psi$ coincides with the action of $\big(-\Delta\big)^s$ on that extension.

When Ω is a ball, since $(-\Delta)^s_{\mathrm{SR}}\psi(x) \in L^\infty(\Omega)$, we have

$$\int_{\mathbb{R}^d} \Big[\psi(x) - \psi\big(\eta(x,v)\big) - \nabla\psi(x)\cdot\big(\eta(x,v)-x\big)\Big]\frac{\mathrm{d}v}{|v|^{d+2s}}$$
$$+ P.V.\int_{\mathbb{R}^d} \nabla\psi(x)\cdot\big(\eta(x,v)-x\big)\frac{\mathrm{d}v}{|v|^{d+2s}}$$

is in $L^\infty(\Omega)$. In the first integral

$$\psi(x) - \psi\big(\eta(x,v)\big) - \nabla\psi(x)\cdot\big(\eta(x,v)-x\big) = O(|x-\eta(x,v)|^2)$$

which means the integral is finite since $2s<2$. Hence, we have

$$\nabla\psi(x)\cdot P.V.\int_{\mathbb{R}^d}\big(\eta(x,v)-x\big)\frac{\mathrm{d}v}{|v|^{d+2s}} \in L^\infty(\Omega) \tag{III.82}$$

Let us show that there is a function $f(x)$ such that

$$P.V.\int_{\mathbb{R}^d}\big(\eta(x,v)-x\big)\frac{\mathrm{d}v}{|v|^{d+2s}} = f(x)n(x) \qquad \text{with} \qquad f(x) \underset{x\to\partial\Omega}{\to} -\infty \tag{III.83}$$

where $n(x)$ denotes the extended outward normal vector: $n(x)=x/|x|$ if $x\neq 0$. We write the integral in a orthonormal coordinates system that starts with $e_1 = n(x)$ and with the notation $\eta(x,v)=\sum \eta_i(x,v)e_i$. We have:

$$\begin{aligned}
P.V.\int_{\mathbb{R}^d}\big(\eta(x,v)-x\big)\frac{\mathrm{d}v}{|v|^{d+2s}} &= \bigg(P.V.\int_{\mathbb{R}^d}\big(\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}}\bigg)e_1 \\
&\quad + \sum_{2\le i\le d}\bigg(P.V.\int_{\mathbb{R}^d}\eta_i(x,v)\frac{\mathrm{d}v}{|v|^{d+2s}}\bigg)e_i \\
&:= I_1 n(x) + \sum_{2\le i\le d} I_i e_i.
\end{aligned}$$

For the coefficient I_2 we notice that if we call $T_2 : y\in\mathbb{R}^d \mapsto y-2y_2e_2$, the mirror image of y with respect to the hyperplane $\{y_2=0\}$, then it is easy to see that the ball is invariant by T_2: $T_2(B_1)=B_1$ which means that η acts in $T_2(B_1)$ exactly as it acts on B_1. As a consequence, T_2 and η commute: $\eta(x,T_2v) = T_2\eta(x,v)$ which yields:

$$\begin{aligned}
I_2 &= \lim_{\varepsilon\to 0} \iint\limits_{\{|v_1|\geq\varepsilon\}\times\mathbb{R}^{d-2}} \left(\int\limits_{v_2>0} \eta_2(x,v)\frac{\mathrm{d}v_2}{|v|^{d+2s}} + \int\limits_{v_2<0} \eta_2(x,v)\frac{\mathrm{d}v_2}{|v|^{d+2s}} \right) \mathrm{d}v_1\,\mathrm{d}v_3\cdots\mathrm{d}v_d \\
&= \lim_{\varepsilon\to 0} \iint\limits_{\{|v_1|\geq\varepsilon\}\times\mathbb{R}^{d-2}} \left(\int\limits_{v_2>0} \eta_2(x,v)\frac{\mathrm{d}v_2}{|v|^{d+2s}} + \int\limits_{v_2>0} (\eta_2(x,Tv))\frac{\mathrm{d}v_2}{|v|^{d+2s}} \right) \mathrm{d}v_1\,\mathrm{d}v_3\cdots\mathrm{d}v_d \\
&= \lim_{\varepsilon\to 0} \iint\limits_{\{|v_1|\geq\varepsilon\}\times\mathbb{R}^{d-2}} \left(\int\limits_{v_2>0} \eta_2(x,v)\frac{\mathrm{d}v_2}{|v|^{d+2s}} + \int\limits_{v_2>0} (-\eta_2(x,v))\frac{\mathrm{d}v_2}{|v|^{d+2s}} \right) \mathrm{d}v_1\,\mathrm{d}v_3\cdots\mathrm{d}v_d \\
&= 0.
\end{aligned}$$

The same holds for all I_i, $i\geq 2$ so that we can define a function $f(x) = I_1$ with which

$$P.V.\int\limits_{\mathbb{R}^d} \big(\eta(x,v)-x\big)\frac{\mathrm{d}v}{|v|^{d+2s}} = \left(P.V.\int\limits_{\mathbb{R}^d} \big(\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}}\right)n(x) := f(x)n(x).$$

To understand the behaviour of f as x goes to the boundary we split the integral as follows, for some $R>0$ fixed, writing $B_{1-|x|}$ the ball centred at 0 of radius $1-|x|$ and C_R the cube centred at 0 of side $2R$ (assuming w.l.o.g. that $R>1-|x|$):

$$\begin{aligned}
f(x) =& P.V.\int\limits_{B_{1-|x|}} \big((\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}} \\
&\qquad + \int\limits_{C_R\setminus B_{1-|x|}} \big(\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}} + \int\limits_{\mathbb{R}^d\setminus C_R} \big(\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}}.
\end{aligned}$$

For the first term on the right-hand-side, we use the explicit expression of η when there are no reflections: $\eta(x,v) = x+v$ in order to write

$$P.V.\int\limits_{B_{1-|x|}} \big((\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}} = \lim_{\varepsilon\to 0}\int\limits_{\varepsilon<|v|<1-|x|} v_1\frac{\mathrm{d}v}{|v|^{d+2s}} = 0$$

because the integrand is an odd function and the domain is radially symmetric. For the last term in the expression of $f(x)$ we write

$$\left|\int\limits_{\mathbb{R}^d\setminus C_R} \big(\eta_1(x,v)-|x|\big)\frac{\mathrm{d}v}{|v|^{d+2s}}\right| \leq \int\limits_{|v|>R} \frac{|v|}{|v|^{d+2s}}\,\mathrm{d}v = \frac{1}{2sR^{2s-1}}$$

which is fixed with R. Finally, for the second term in the expression of $f(x)$, we want to identify a sign in the integrand to which end we introduce

$$\mathcal{E}(x) = \Big(C_R \setminus B_{1-|x|}\Big) \cap \Big(\big\{ -R \le v_1 \le 0\big\} \cup \big\{2(1-|x|) \le v_1 \le R\big\}\Big)$$

so that for any $v \in \mathcal{E}(x)$ we have $\eta_1(x,v) - |x| \le 0$ (note that the set of all velocities such that $\eta_1(x,v) - |x| \le 0$ is actually a little bigger that $\mathcal{E}(x)$ because of the curvature of $\partial\Omega$, if $\partial\Omega$ was a straight line that it would be precisely $\mathcal{E}(x)$). We also write $\mathcal{E}^c(x) = (C_R \setminus B_{1-|x|}) \setminus \mathcal{E}(x)$ its complement in $C_R \setminus B_{1-|x|}$ with which we have

$$\int\limits_{C_R \setminus B_{1-|x|}} \big(\eta_1(x,v) - |x|\big) \frac{\mathrm{d}v}{|v|^{d+2s}} = -\int\limits_{\mathcal{E}(x)} \big|\eta_1(x,v) - x\big| \frac{\mathrm{d}v}{|v|^{d+2s}} + \int\limits_{\mathcal{E}^c(x)} \big(\eta_1(x,v) - x\big) \frac{\mathrm{d}v}{|v|^{d+2s}}.$$

We introduce the notations

$$\mathcal{E}(x,v_1) = \Big\{ (v_2,\cdots,v_d) : \sqrt{(1-|x|)^2 - v_1^2} \le |v_2|,\cdots,|v_d| \le R \Big\}$$

and

$$\mathcal{E}^c(x,v_1) = \Big\{ (v_2,\cdots,v_d) : \sqrt{(1-x)^2 - v_1^2} \le |v_2|,\cdots,|v_d| \le 2(1-|x|) \Big\}$$

such that for a fixed v_1 the projection of $\mathcal{E}(x)$ on $\{w \in \mathbb{R}^d : w_1 = v_1\}$ is $\mathcal{E}(x,v_1)$ if $-R \le v_1 \le 0$ and $[-R,R]^{d-1}$ if $2(1-|x|) \le v_1 \le R$, and the projection of $\mathcal{E}^c(x)$ is $\mathcal{E}^c(x,v_1)$. With those, we have on the one hand

$$\begin{aligned}\int\limits_{\mathcal{E}(x)} \big|\eta_1(x,v) - x\big| \frac{\mathrm{d}v}{|v|^{d+2s}} &= \int\limits_{v_1=-R}^{|x|} \Bigg(\int\limits_{\mathcal{E}(x,v_1)} \frac{\big|\eta_1(x,v) - x\big|}{|v|^{d+2s}} \,\mathrm{d}v_2 \cdots \mathrm{d}v_d \Bigg) \mathrm{d}v_1 \\ &+ \int\limits_{v_1=2(1-|x|)}^{R} \Bigg(\int\limits_{[-R,R]^{d-1}} \frac{\big|\eta_1(x,v) - x\big|}{|v|^{d+2s}} \,\mathrm{d}v_2 \cdots \mathrm{d}v_d \Bigg) \mathrm{d}v_1\end{aligned}$$

and on the other hand

$$\Bigg| \int\limits_{\mathcal{E}^c(x)} \big(\eta_1(x,v) - x\big) \frac{\mathrm{d}v}{|v|^{d+2s}} \Bigg| \le \int\limits_{v_1=0}^{2(1-|x|)} \Bigg(\int\limits_{\mathcal{E}^c(x,v_1)} \frac{\big|\eta_1(x,v) - x\big|}{|v|^{d+2s}} \,\mathrm{d}v_2 \cdots \mathrm{d}v_d \Bigg) \mathrm{d}v_1.$$

We see that it is the same integrand but in the integral over $\mathcal{E}^c(x)$, the volume of the domain of integration $\big(0, 2(1-|x|)\big)\times\mathcal{E}^c(x, v_1)$ goes to 0 as x approaches the boundary whereas the domains $(-R, |x|)\times\mathcal{E}(x, v_1)$ and $\big(2(1-|x|), R\big)\times(-R, R)^2$ do not, hence the first term is negligible in the limit before the second and we have

$$
\begin{aligned}
|f(x)| \underset{x\to\partial\Omega}{\sim} \int_{\mathcal{E}(x)} \big|\eta_1(x,v) - x\big| \frac{\mathrm{d}v}{|v|^{d+2s}} &\geq \int_{v_1=-R}^{|x|} \left(\iint_{\mathcal{E}(x,v_1)} \frac{v_1}{|v|^{d+2s}} \,\mathrm{d}v_2 \,\mathrm{d}v_3 \right) \mathrm{d}v_1 \\
&\geq \iint_{1-|x|\leq|v_2|,|v_3|\leq R} \left(\int_{-R}^{|x|-1} \frac{v_1}{|v|^{d+2s}} \right) \mathrm{d}v_2 \,\mathrm{d}v_3 \\
&\geq \iint_{1-|x|\leq|v_2|,|v_3|\leq R} \frac{C}{\Big((1-|x|)^2 + v_2^2 + v_3^2\Big)^{(d+2s-2)/2}} \,\mathrm{d}v_2 \,\mathrm{d}v_3.
\end{aligned}
$$

As x approaches the boundary, the integrand tends to $1/(v_2^2 + v_3^2)^{d-1+2s-1}$ and the domain to $[-R, R]^{d-1}$ so the integral diverges since $2s - 1 > 0$. $\square$

Chapter IV

Classical diffusion limit in spatially bounded domains

Joint work with Harsha Hutridurga

Diffusion limit for Vlasov-Fokker-Planck equation in bounded domains arXiv:1604.08388 (2016).

Contents

IV.1 Introduction

IV.1.1 The Vlasov-Fokker-Planck equation

In this paper, we study the diffusion limit of Vlasov-Fokker-Planck equation in a bounded spatial domain with specular reflections on the boundary. The equation we consider models the behavior of a low density gas in the absence of macroscopic force field. Introducing the probability density $f(t,x,v)$, i.e., the probability of finding a particle with velocity v at time t and position x, we consider the evolution equation

$$\partial_t f + v\cdot\nabla_x f = \mathcal{L}f := \nabla_v\cdot\Big(\nabla_v f + vf\Big) \qquad \text{for } (t,x,v)\in(0,T)\times\Omega\times\mathbb{R}^d, \quad \text{(IV.1a)}$$

$$f(0,x,v) = f^{in}(x,v) \qquad \text{for } (x,v)\in\Omega\times\mathbb{R}^d. \quad \text{(IV.1b)}$$

The left hand side of (IV.1a) models the free transport of particles, while the Fokker-Planck operator $\mathcal{L}$ on the right hand side describes the interaction of the particles with the background. It can be interpreted as a deterministic description of a Langevin equation for the velocity of the particles:

$$\dot{v}(t) = -\nu v(t) + W(t),$$

where the friction coefficient ν will be assumed, without loss of generality, equal to 1 and $W(t)$ is a Gaussian white noise. We consider (IV.1a) on a smooth bounded domain $\Omega\subset\mathbb{R}^d$ in the sense that there exists a smooth function $\zeta:\mathbb{R}^d\mapsto\mathbb{R}$ such that

$$\Omega = \{x\in\mathbb{R}^d \text{ s.t. } \zeta(x)<0\}; \qquad \partial\Omega = \{x\in\mathbb{R}^d \text{ s.t. } \zeta(x)=0\}. \qquad \text{(IV.2)}$$

In order to define a normal vector at each point on the boundary we assume that $\nabla_x\zeta(x)\neq 0$ for any x such that $\zeta(x)\ll 1$ and we define the unit outward normal vector, for any $x\in\partial\Omega$, as

$$n(x) := \frac{\nabla_x\zeta(x)}{|\nabla_x\zeta(x)|}.$$

Moreover, we also assume that Ω is strongly convex, namely that there exists a constant $C_\zeta>0$ such that

$$\sum_{i,j=1}^{d}\xi_i\frac{\partial^2\zeta}{\partial x_i\partial x_j}\xi_j \geq C_\zeta|\xi|^2 \qquad \forall\,\xi\in\mathbb{R}^d. \qquad \text{(IV.3)}$$

To define boundary conditions in the phase space, we introduce the following notations:

$$\begin{aligned}
&\Sigma := \{(x,v) \in \partial\Omega \times \mathbb{R}^d\} && \text{Phase space Boundary,}\\
&\Sigma_+ := \{(x,v) \in \partial\Omega \times \mathbb{R}^d \text{ such that } v \cdot n(x) > 0\} && \text{Outgoing Boundary,}\\
&\Sigma_- := \{(x,v) \in \partial\Omega \times \mathbb{R}^d \text{ such that } v \cdot n(x) < 0\} && \text{Incoming Boundary,}\\
&\Sigma_0 := \{(x,v) \in \partial\Omega \times \mathbb{R}^d \text{ such that } v \cdot n(x) = 0\} && \text{Grazing set.}
\end{aligned}$$

We denote by γf the trace of f on Σ. Boundary conditions for (IV.1a) take the form of a balance law between the traces of f on Σ_+ and Σ_- which we denote by $\gamma_+ f$ and $\gamma_- f$ respectively. We shall consider, throughout this paper, the specular reflection boundary condition which is illustrated in Figure IV.1 and reads

$$\gamma_- f(t,x,v) = \gamma_+ f(t,x,\mathcal{R}_x v) \quad \text{for } (t,x,v) \in (0,T) \times \Sigma_-, \tag{IV.4}$$

where $\mathcal{R}_x$ is the reflection operator on the space of velocities given by

$$\mathcal{R}_x v := v - 2\big(v \cdot n(x)\big)n(x).$$

Note that this reflection operator changes the direction of the velocity at the boundary but it preserves the magnitude, i.e., $|\mathcal{R}_x v| = |v|$.

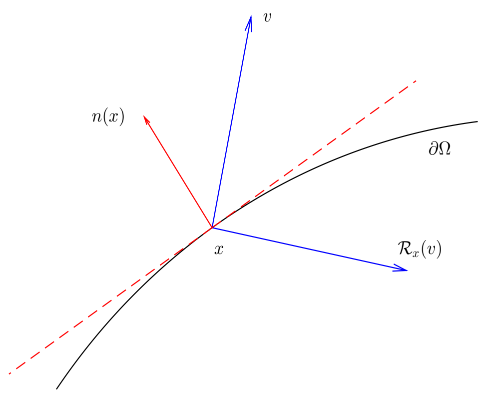


Fig. IV.1 Specular reflection operator

IV.1.2 Main result

In order to investigate the diffusion limit of (IV.1a)-(IV.1b), we introduce the Knudsen number $0 < \varepsilon \ll 1$ which represents the ratio of the mean free path to the macroscopic length scale, or equivalently the ratio of the mean time between two kinetic interactions to the macroscopic time scale. We rescale time as $t' = \varepsilon t$ and also introduce a coefficient ε^{-1} in front of the Fokker-Planck operator in (IV.1a) to model the number of collision per unit of time going to infinity. The rescaled equation, thus becomes

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \nabla_v \cdot (v f^\varepsilon + \nabla_v f^\varepsilon) \quad \text{for } (t,x,v) \in (0,T) \times \Omega \times \mathbb{R}^d, \tag{IV.5a}$$

$$f^\varepsilon(0,x,v) = f^{in}(x,v) \quad \text{for } (x,v) \in \Omega \times \mathbb{R}^d, \tag{IV.5b}$$

$$\gamma_- f^\varepsilon(t,x,v) = \gamma_+ f^\varepsilon(t,x,\mathcal{R}_x v) \quad \text{for } (t,x,v) \in (0,T) \times \Sigma_-. \tag{IV.5c}$$

In this paper, we investigate the behavior of the solution f^ε in the $\varepsilon \to 0$ limit. The characterization of the asymptotic behavior of $f^\varepsilon(t,x,v)$ is the object of our main result.

Theorem IV.1.1. *Assume the initial datum $f^{in}(x,v)$ satisfies*

$$f^{in}(x,v) \geq 0 \quad \forall (x,v) \in \Omega \times \mathbb{R}^d; \qquad f^{in} \in \mathrm{L}^2\Big(\Omega \times \mathbb{R}^d, \mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v\Big),$$

where $\mathcal{M}(v)$ is the centered Gaussian

$$\mathcal{M}(v) := \frac{1}{(2\pi)^{d/2}} e^{\frac{-|v|^2}{2}}. \tag{IV.6}$$

Let $f^\varepsilon(t,x,v)$ be a weak solution to the initial boundary value problem (IV.5a)-(IV.5b)-(IV.5c)*. Then*

$$f^\varepsilon(t,x,v) \rightharpoonup \rho(t,x)\mathcal{M}(v) \quad \textit{in } \mathrm{L}^\infty\Big(0,T; \mathrm{L}^2\Big(\Omega \times \mathbb{R}^d, \mathcal{M}^{-1}(v)\mathrm{d}v\mathrm{d}x\Big)\Big) \ \textit{weak-}*$$

as $\varepsilon \to 0$, for some $\rho \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$. Furthermore, if the spatial domain Ω is a ball in $\mathbb{R}^d$ then the limit $\rho(t,x)$ is a weak solution to the diffusion equation

$$\partial_t \rho - \Delta_x \rho = 0 \quad \textit{for } (t,x) \in (0,T) \times \Omega, \tag{IV.7a}$$

$$\rho(0,x) = \rho^{in}(x) \quad \textit{for } x \in \Omega, \tag{IV.7b}$$

$$\nabla_x \rho(t,x) \cdot n(x) = 0 \quad \textit{for } (t,x) \in (0,T) \times \partial\Omega, \tag{IV.7c}$$

with the initial datum

$$\rho^{in}(x) = \int_{\mathbb{R}^d} f^{in}(x,v)\,\mathrm{d}v.$$

IV.1.3 Plan of the paper

Section IV.2 gives some heuristics with regard to the strategy of proof for Theorem IV.1.1. In particular, we compare our method of proof with some standard techniques used to prove the diffusion limit for the kinetic Fokker-Planck equation. In section IV.3, we define an appropriate notion of weak solution to our initial boundary value problem. In section IV.4, we develop the theory of constructing a special class of test functions using an auxiliary problem. Finally, in section IV.5, we arrive at the parabolic limit equation, thus proving Theorem IV.1.1. In the Appendix, and especially section A.0.5 we give regularity results associated with some Hamiltonian dynamics that we will need to study the aforementioned auxiliary problem.

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IV.2 Strategy of the proof

In this section, we lay out the strategy of our proof for Theorem IV.1.1. We would like to demonstrate the novelty in our approach by citing some comparisons with the standard techniques used in the diffusion approximation for the Vlasov-Fokker-Planck equation. Those techniques were introduced in 1987 by Degond and Mas-Gallic [DMG87] for the one dimensional case in bounded domains. They were later improved, in 2000, by Poupaud and Soler [PS00] where they consider the more complicated Vlasov-Poisson-Fokker-Planck equation on the whole space and established the diffusion limit for a small enough time interval. More recently, improving the result further, Goudon [Gou05] established in 2005 the global-in-time convergence in dimension 2 with bounds on the entropy and energy of the initial data so as to ensure that singularities do not develop in the limit system and finally, in 2010, El Ghani and

Masmoudi proved in [EGM10] the global-in-time convergence in higher dimensions with similar initial bounds.
In these papers, the analysis with regard to this nonlinear model is quite involved. Let us simply present the analysis in [PS00] adapted to the linear model (IV.5a). The idea is to consider the continuity equation for the local densities $\rho^\varepsilon(t,x)$ given by

$$\frac{\partial \rho^\varepsilon}{\partial t}(t,x) + \frac{1}{\varepsilon}\nabla_x \cdot j^\varepsilon(t,x) = 0,$$

where the current density $j^\varepsilon(t,x)$ is defined as

$$j^\varepsilon(t,x) := \int\limits_{\mathbb{R}^d} v f^\varepsilon(t,x,v)\,\mathrm{d}v.$$

The principal idea is to obtain

$$\begin{aligned} &\rho^\varepsilon \rightharpoonup \rho \qquad \text{weakly in } \mathrm{L}^1((0,T)\times\Omega), \\ &\frac{1}{\varepsilon} j^\varepsilon \rightharpoonup \nabla_x \rho \qquad \text{in } \mathcal{D}'((0,T)\times\Omega) \end{aligned} \tag{IV.8}$$

as $\varepsilon \to 0$. The article [PS00] is concerned with the analysis in the full spatial domain $\mathbb{R}^d$. In order to derive the limit boundary condition – we refer the interested reader to the paper [WLL15b] of Wu, Lin and Liu for more details – one can multiply the specular reflection boundary condition (IV.5c) by $(v\cdot n(x))$ and integrate over the incoming velocities at the point $x \in \partial\Omega$ yielding

$$\int\limits_{v\cdot n(x)<0} \gamma_- f^\varepsilon(t,x,v)\,(v\cdot n(x))\;\mathrm{d}v = \int\limits_{v\cdot n(x)<0} \gamma_+ f^\varepsilon(t,x,\mathcal{R}_x(v))\,(v\cdot n(x))\;\mathrm{d}v.$$

Making the change of variables $w = \mathcal{R}_x(v)$ on the right hand side of the above expression yields

$$\int\limits_{v\cdot n(x)<0} \gamma_- f^\varepsilon(t,x,v)\,(v\cdot n(x))\;\mathrm{d}v = -\int\limits_{w\cdot n(x)>0} \gamma_+ f^\varepsilon(t,x,w)\,(w\cdot n(x))\;\mathrm{d}w.$$

This implies the following

$$\int\limits_{\mathbb{R}^d} \gamma f^\varepsilon\,(v\cdot n(x))\;\mathrm{d}v = 0 \implies j^\varepsilon(t,x)\cdot n(x) = 0.$$

Taking the limits (IV.8) into consideration, we do have the homogeneous Neumann condition on the boundary in the $\varepsilon \to 0$ limit.
Our strategy is essentially different in the sense that we exploit the hyperbolic structure of the Vlasov-Fokker-Planck equation that appears in Fourier space, as we will explain in section IV.4, and which reveals, when coupled with the reflective boundaries, the underlying Hamiltonian dynamics of the kinetic equation. We will take advantage of the dynamics by constructing a special class of test functions for the weak formulation (IV.11) of the initial boundary value problem (IV.5a)-(IV.5c)-(IV.5b) and then passing to the limit for such test functions, only using the weak L^2-compactness result (see Proposition IV.3.2).

IV.2.1 Efficiency of our approach

To justify the interest of our method, we prove the diffusion limit in full space, i.e., when $\Omega = \mathbb{R}^d$. It only takes a few lines which shows how efficient our method is in the Fokker-Planck context. Let us consider the scaled (diffusive scaling) Vlasov-Fokker-Planck equation for the probability density $f^\varepsilon(t,x,v)$ in the full space.

$$\begin{aligned} &\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \nabla_v \cdot \Big(\nabla_v f^\varepsilon + v f^\varepsilon \Big) && \text{for } (t,x,v) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d, \\ &f^\varepsilon(0,x,v) = f^{in}(x,v) && \text{for } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{aligned}$$

This equation has a unique weak solution $f^\varepsilon(t,x,v)$ which satisfies

$$f^\varepsilon \in \mathrm{L}^2\big((0,T)\times\mathbb{R}^d_x; \mathrm{H}^1(\mathbb{R}^d_v)\big) \text{ and } \partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon \in \mathrm{L}^2\big((0,T)\times\mathbb{R}^d_x; \mathrm{H}^{-1}(\mathbb{R}^d_v)\big)$$

as was proven by Degond in the appendix of [Deg86]. Moreover, the Fokker-Planck operator is dissipative in the sense that

$$-\iint_{\mathbb{R}^d\times\mathbb{R}^d} f^\varepsilon \mathcal{L}\big(f^\varepsilon\big) \frac{\mathrm{d}x\mathrm{d}v}{\mathcal{M}(v)} \geq 0$$

from which, as we will prove in section IV.3, we can show that f^ε converges weakly$*$ in $\mathrm{L}^\infty\left(0,T;\mathrm{L}^2\left(\Omega\times\mathbb{R}^d, \mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v\right)\right)$ to $\rho(t,x)\mathcal{M}(v)$ where $\rho(t,x)$ is the limit of the local densities $\rho^\varepsilon(t,x) := \int_{\mathbb{R}^d} f^\varepsilon \,\mathrm{d}v$.

For any $\psi \in \mathcal{C}_c^\infty((0,T)\times\mathbb{R}^d)$ we construct the test function $\phi^\varepsilon(t,x,v) = \varphi(t, x+\varepsilon v)$ with which the weak formulation of the Vlasov-Fokker-Planck equation reads

$$\iiint\limits_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} f^\varepsilon(t,x,v)\Big(\varepsilon^2\partial_t\phi^\varepsilon + \varepsilon v\cdot\nabla_x\phi^\varepsilon + \Delta_v\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon\Big)(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\,\varepsilon^2\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\phi^\varepsilon(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

Our particular choice of the test functions enables us to have

$$\varepsilon v\cdot\nabla_x\phi^\varepsilon = v\cdot\nabla_v\phi^\varepsilon \qquad \text{and} \qquad \Delta_v\phi^\varepsilon = \varepsilon^2\Delta_x\phi^\varepsilon.$$

Thus, we have

$$\iiint\limits_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} f^\varepsilon(t,x,v)\Big(\partial_t\varphi + \Delta_x\varphi\Big)(t,x+\varepsilon v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$
$$+\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} f^{in}(x,v)\varphi(0,x+\varepsilon v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

Passing to the limit in the above expression as $\varepsilon \to 0$, using the weak convergence of f^ε and the regularity of φ with respect to both its variables, yields

$$\iint\limits_{(0,T)\times\mathbb{R}^d} \rho(t,x)\Big(\partial_t\varphi + \Delta_x\varphi\Big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\mathbb{R}^d}\rho^{in}(x)\varphi(0,x)\,\mathrm{d}x = 0$$

which is the weak formulation of

$$\begin{aligned} &\partial_t\rho - \Delta_x\rho = 0 && \text{for } (t,x)\in(0,T)\times\mathbb{R}^d,\\ &\rho(0,x) = \rho^{in}(x) && \text{for } x\in\mathbb{R}^d. \end{aligned}$$

IV.3 Solutions of the Vlasov-Fokker-Planck equation

Several works from the 80's and 90's investigate the existence of solution to the Vlasov-Fokker-Planck equation. We refer the interested reader to [Deg86] for the global existence of smooth solution in the whole space in space dimensions 1 and 2 and to [Car98] for global weak solutions on a bounded domain with absorbing-type boundary condition. More recently, Mellet and Vasseur established existence of global weak solution with reflection-law on the boundary in [MV07].

IV.3.1 Existence of weak solution

The present work is in a very similar framework and we will therefore use the same kind of definition for weak solution as in [MV07].

Definition IV.3.1. *We say that $f(t,x,v)$ is a weak solution of* (IV.1a)-(IV.1b)-(IV.4) *on* $[0,T]$ *if*

$$\begin{aligned} &f(t,x,v) \geq 0 \qquad \forall (t,x,v) \in [0,T]\times\Omega\times\mathbb{R}^d, \\ &f \in C\left([0,T];\mathrm{L}^1(\Omega\times\mathbb{R}^d)\right) \cap \mathrm{L}^\infty\left(0,T;\mathrm{L}^1\cap\mathrm{L}^\infty(\Omega\times\mathbb{R}^d)\right) \end{aligned} \tag{IV.9}$$

and (IV.1a) *holds in the sense that for any $\phi(t,x,v)$ such that*

$$\begin{aligned} &\phi \in C^\infty\left([0,T]\times\overline{\Omega}\times\mathbb{R}^d\right), \qquad \phi(T,\cdot,\cdot) = 0, \\ &\gamma_+\phi(t,x,v) = \gamma_-\phi\left(t,x,\mathcal{R}_x(v)\right) \quad \forall (t,x,v)\in[0,T]\times\Sigma_+, \end{aligned} \tag{IV.10}$$

we have

$$\begin{aligned} \iiint\limits_{(0,T)\times\Omega\times\mathbb{R}^d} f(t,x,v)\Big(\partial_t\phi + v\cdot\nabla_x\phi - v\cdot\nabla_v\phi + \Delta_v\phi\Big)(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ + \iint\limits_{\Omega\times\mathbb{R}^d} f^{in}(x,v)\phi(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0. \end{aligned} \tag{IV.11}$$

Such a definition is required as it is well-known for kinetic equations that the specular reflection condition causes a loss in regularity of the solution, in comparison with absorption type boundary condition, as is explained in detail in [Mis10]. Hence, we introduce the above formulation where the boundary condition is satisfied in a weak sense. With such a notion of weak solution, we have the following result of existence from [MV07].

Theorem IV.3.1. *Let the initial data $f^{in}(x,v)$ satisfy*

$$f^{in}(x,v) \geq 0 \quad \forall (x,v)\in\Omega\times\mathbb{R}^d; \qquad f^{in} \in \mathrm{L}^2(\Omega\times\mathbb{R}^d, \mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v). \tag{IV.12}$$

Then there exists a weak solution to (IV.1a)-(IV.1b) *satisfying* (IV.4) *defined globally-in-time. Moreover, we have the a priori estimate*

$$\sup_{t\in(0,T)} \iint_{\Omega\times\mathbb{R}^d} |f(t,x,v)|^2 \frac{\mathrm{d}x\mathrm{d}v}{\mathcal{M}(v)} + \int_0^T \mathcal{D}(f)(t)\,\mathrm{d}t \le \iint_{\Omega\times\mathbb{R}^d} |f^{in}(x,v)|^2 \frac{\mathrm{d}x\mathrm{d}v}{\mathcal{M}(v)}, \tag{IV.13}$$

where the dissipation $\mathcal{D}$ is given by:

$$\mathcal{D}(f) = -2 \iint_{\Omega\times\mathbb{R}^d} f(t,x,v)\mathcal{L}f(t,x,v) \frac{\mathrm{d}x\mathrm{d}v}{\mathcal{M}(v)}. \tag{IV.14}$$

The proof of the above theorem is similar to the proof of Theorem 2.2 in [MV07]. It consists of approximating the specular reflection condition (IV.4) through induction on Dirichlet boundary conditions and showing that regularity (IV.9) and estimates (IV.13) hold as we pass to the limit in the induction procedure. As it is not the principal focus of this article, we will not give a detailed proof of Theorem IV.3.1. However, in an effort to motivate the estimate (IV.13), we present the following, rather formal, computation.

Assume f has a trace in $\mathrm{L}^2(0,T;\mathrm{L}^2(\Sigma_+))$. Multiply (IV.1a) by $\mathcal{M}^{-1}(v)f(t,x,v)$ and integrate over the phase space $\Omega\times\mathbb{R}^d$ yielding

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega\times\mathbb{R}^d} |f(t,x,v)|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{\mathcal{M}(v)} + \iint_{\Omega\times\mathbb{R}^d} v\cdot\nabla_x \left(f(t,x,v)\right)^2 \frac{\mathrm{d}v\,\mathrm{d}x}{\mathcal{M}(v)} \\ = 2\iint_{\Omega\times\mathbb{R}^d} \mathcal{L}f(t,x,v)f(t,x,v) \frac{\mathrm{d}v\,\mathrm{d}x}{\mathcal{M}(v)}. \end{aligned} \tag{IV.15}$$

For the second term on the left hand side of the above expression, using the assumption on the trace of f, we write

$$\begin{aligned} &\iint_{\Omega\times\mathbb{R}^d} v\cdot\nabla_x \left(f(t,x,v)\right)^2 \frac{\mathrm{d}v\,\mathrm{d}x}{\mathcal{M}(v)} = \iint_{\Sigma} |\gamma f|^2 \left(v\cdot n(x)\right) \frac{\mathrm{d}v\,\mathrm{d}x}{\mathcal{M}(v)} \\ &= \iint_{\Sigma_+} |\gamma_+ f(t,x,v)|^2 \,|v\cdot n(x)|\, \frac{\mathrm{d}v\,\mathrm{d}\sigma(x)}{\mathcal{M}(v)} - \iint_{\Sigma_-} |\gamma_- f(t,x,v)|^2 \,|v\cdot n(x)|\, \frac{\mathrm{d}v\,\mathrm{d}\sigma(x)}{\mathcal{M}(v)} \end{aligned}$$

where, using the specular reflection (IV.4) and the fact that $\mathcal{M}(v)$ is radial, the change of variable $w = \mathcal{R}_x(v)$ yields

$$\iint\limits_{\Sigma_-} |\gamma_- f(t,x,v)|^2 \, |v \cdot n(x)| \, \frac{\mathrm{d}v \, \mathrm{d}\sigma(x)}{\mathcal{M}(v)} = \iint\limits_{\Sigma_+} |\gamma_+ f(t,x,w)|^2 \, |w \cdot n(x)| \, \frac{\mathrm{d}w \, \mathrm{d}\sigma(x)}{\mathcal{M}(w)}.$$

This implies that the second term on the left hand side of the expression (IV.15) does not contribute. Hence, we arrive at the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint\limits_{\Omega\times\mathbb{R}^d} |f(t,x,v)|^2 \, \frac{\mathrm{d}v \, \mathrm{d}x}{\mathcal{M}(v)} = -\mathcal{D}(f).$$

Integrating the above identity over the time interval $(0,T)$ yields the a priori estimate (IV.13).

IV.3.2 Uniform a priori estimate

The notion of weak solution (Definition IV.3.1) and the theorem of existence (Theorem IV.3.1) hold for the scaled equation (IV.5a)-(IV.5b)-(IV.5c) for any $\varepsilon > 0$. The scaling only changes the estimate (IV.13) which becomes

$$\sup_{t\in(0,T)} \iint\limits_{\Omega\times\mathbb{R}^d} |f^\varepsilon(t,x,v)|^2 \, \frac{\mathrm{d}v \, \mathrm{d}x}{\mathcal{M}(v)} + \frac{1}{\varepsilon^2} \int_0^T \mathcal{D}(f^\varepsilon)(t) \, \mathrm{d}t \leq \iint\limits_{\Omega\times\mathbb{R}^d} |f^{in}(x,v)|^2 \, \frac{\mathrm{d}v \, \mathrm{d}x}{\mathcal{M}(v)} \tag{IV.16}$$

as one can formally see by doing the computation involving (IV.15) with the scaling. We shall use the estimate (IV.16) to prove the following result.

Proposition IV.3.2. *Let $f^\varepsilon(t,x,v)$ be a weak solution of the scaled Vlasov-Fokker-Planck equation with specular reflection* (IV.5a)-(IV.5b)-(IV.5c) *in the sense of Definition IV.3.1 with an initial datum $f^{in}(x,v)$ which satisfies* (IV.12). *Then there exists $\rho \in L^2((0,T)\times\Omega)$ such that*

$$f^\varepsilon \rightharpoonup \rho(t,x)\mathcal{M}(v) \quad \text{weakly in } \mathrm{L}^2\left(0,T;\mathrm{L}^2\left(\Omega\times\mathbb{R}^d, \mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v\right)\right) \tag{IV.17}$$

where $\rho(t,x)$ is the weak- limit of the local densities*

$$\rho^\varepsilon(t,x) := \int\limits_{\mathbb{R}^d} f^\varepsilon(t,x,v) \, \mathrm{d}v \tag{IV.18}$$

in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))$.

Proof. The proof relies on the properties of the dissipation (IV.14) in the estimate (IV.16). Remark that the Fokker-Planck operator can be rewritten as

$$\mathcal{L}f^\varepsilon = \nabla_v \cdot \left(\mathcal{M}(v)\nabla_v\left(\frac{f^\varepsilon}{\mathcal{M}(v)}\right)\right).$$

This helps us deduce that the dissipation $\mathcal{D}$ is positive semi-definite, i.e.,

$$\mathcal{D}(f^\varepsilon) = -\iint\limits_{\Omega\times\mathbb{R}^d} \frac{f^\varepsilon}{\mathcal{M}(v)}\mathcal{L}f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x = \iint\limits_{\Omega\times\mathbb{R}^d} \left|\nabla_v\left(\frac{f^\varepsilon}{\mathcal{M}(v)}\right)\right|^2 \mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}x \geq 0.$$

The non-negativity of $\mathcal{D}$ in (IV.16) yields the following uniform (with respect to ε) bound.

$$\|f^\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega\times\mathbb{R}^d,\mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v))} \leq C. \tag{IV.19}$$

Hence, we can extract a sub-sequence and there exists a limit $\overline{f}(t,x,v)$ such that

$$f^\varepsilon \rightharpoonup \overline{f} \quad \text{in } \mathrm{L}^\infty\left(0,T;\mathrm{L}^2\left(\Omega\times\mathbb{R}^d,\mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v\right)\right) \text{ weak-*}$$

as $\varepsilon \to 0$. Moreover, using the Cauchy-Schwarz inequality, we have

$$|\rho^\varepsilon(t,x)| = \left|\int\limits_{\mathbb{R}^d} \frac{f^\varepsilon}{\mathcal{M}^{1/2}}\mathcal{M}^{1/2}\,\mathrm{d}v\right| \leq \left(\int\limits_{\mathbb{R}^d} |f^\varepsilon|^2\frac{\mathrm{d}v}{\mathcal{M}(v)}\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}^d}\mathcal{M}(v)\,\mathrm{d}v\right)^{\frac{1}{2}}.$$

Since $\mathcal{M}(v)$ is normalized (IV.6), integrating the above inequality in the spatial variable and taking supremum over the time interval $[0,T]$ yields the following estimate

$$\|\rho^\varepsilon\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^2(\Omega))} \leq C, \tag{IV.20}$$

where we have used the estimate (IV.19). Again, we can extract a sub-sequence and there exists a limit $\rho(t,x)$ such that

$$\rho^\varepsilon \rightharpoonup \rho \quad \text{in } \mathrm{L}^\infty\left(0,T;\mathrm{L}^2(\Omega)\right) \text{ weak-*}.$$

Remark that the dissipation can be successively written as

$$\mathcal{D}(f^\varepsilon) = \iint\limits_{\Omega\times\mathbb{R}^d} \left|\nabla_v \left(\frac{f^\varepsilon}{\mathcal{M}(v)}\right)\right|^2 \mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}x = \iint\limits_{\Omega\times\mathbb{R}^d} \left|\nabla_v \left(\frac{f^\varepsilon - \rho^\varepsilon\mathcal{M}(v)}{\mathcal{M}(v)}\right)\right|^2 \mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}x.$$

Using Poincaré inequality for the Gaussian measure in the velocity variable yields the existence of a constant $\theta > 0$ such that

$$\mathcal{D}(f^\varepsilon) \geq \theta \iint\limits_{\Omega\times\mathbb{R}^d} \left|\frac{f^\varepsilon - \rho^\varepsilon\mathcal{M}(v)}{\mathcal{M}(v)}\right|^2 \mathcal{M}(v)\,\mathrm{d}v\,\mathrm{d}x = \theta \iint\limits_{\Omega\times\mathbb{R}^d} |f^\varepsilon - \rho^\varepsilon\mathcal{M}(v)|^2 \frac{\mathrm{d}v\mathrm{d}x}{\mathcal{M}(v)}.$$

Since (IV.16) implies that the dissipation tends to zero as ε tends to zero, we have

$$f^\varepsilon - \rho^\varepsilon\mathcal{M}(v) \to 0 \quad \text{strongly in } \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega\times\mathbb{R}^d, \mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v)).$$

This concludes the proof. □

IV.4 Auxiliary problem

The auxiliary problem that we consider is inspired by the hyperbolic structure of the Vlasov-Fokker-Planck equation in Fourier space. Indeed, if we consider (IV.5a) in the whole space and apply Fourier transform in x and v variables (with respective Fourier variables p and q), we have

$$\varepsilon\partial_t \widehat{f^\varepsilon} + \left(p - \frac{1}{\varepsilon}q\right)\cdot\nabla_q \widehat{f^\varepsilon} = \frac{1}{\varepsilon}|q|^2 \widehat{f^\varepsilon}$$

which is a hyperbolic equation, its characteristic lines given by $(p - \varepsilon^{-1}q)\cdot\nabla_q$. The motivation behind the auxiliary problem is to choose a test function which will be constant along those lines (translated in an adequate way to the non-Fourier space) and satisfy the specular reflection condition (IV.4). This auxiliary problem was first introduced in [CMT12] in the whole space and then improved in [Ces16] to handle bounded domains and in particular specular reflection boundary conditions. For the sake of completeness, let us present the construction of a solution to this problem in a strongly convex domain with specular reflections on the boundary.

IV.4.1 Geodesic Billiards and Specular cycles

For any $\psi \in \mathcal{C}^\infty(\overline{\Omega})$ we construct $\varphi(x,v)$ through the following boundary value problem.

$$\begin{cases} v \cdot \nabla_x \varphi - v \cdot \nabla_v \varphi = 0 & \text{in } \Omega \times \mathbb{R}^d, \\ \gamma_+ \varphi(x,v) = \gamma_- \varphi(x, \mathcal{R}_x(v)) & \text{on } \Sigma_+, \\ \varphi(x,0) = \psi(x) & \text{in } \Omega, \end{cases} \tag{IV.21}$$

where we impose the initial condition on the hypersurface $\{v = 0\}$. Note that $\phi^\varepsilon(x,v) = \varphi(x, \varepsilon v)$ will be a solution to the following auxiliary problem.

$$\begin{cases} \varepsilon v \cdot \nabla_x \phi^\varepsilon - v \cdot \nabla_v \phi^\varepsilon = 0 & \text{in } \Omega \times \mathbb{R}^d, \\ \gamma_+ \phi^\varepsilon(x,v) = \gamma_- \phi^\varepsilon(x, \mathcal{R}_x(v)) & \text{on } \Sigma_+, \\ \phi^\varepsilon(x,0) = \psi(x) & \text{in } \Omega. \end{cases} \tag{IV.22}$$

The characteristic curves associated with the boundary value problem (IV.21) solve the following system of ordinary differential equations.

$$\begin{cases} \dot{x}(s) = v(s) & x(0) = x_0, \\ \dot{v}(s) = -v(s) & v(0) = v_0, \\ \text{If } x(s) \in \partial\Omega \text{ then } v(s^+) = \mathcal{R}_{x(s)}(v(s^-)). & \end{cases} \tag{IV.23}$$

We denote by $\Psi_{x_0,v_0}(s) = (x(s), v(s))$ to be the flow associated with (IV.23) in the phase space $\Omega \times \mathbb{R}^d$ starting at (x_0, v_0). Suppose the base point of the flow is an arbitrary $(x_0, v_0) \in \Omega \times \mathbb{R}^d$. With the convention $s_0 = 0$, consider the sequence $\{s_i\}_{i \geq 0} \subset [0, \infty)$ of forward exit times defined as

$$s_{i+1}(x_0, v_0) := \inf \Big\{ \ell \in [s_i, \infty) \text{ s.t. } x(s_i) + (\ell - s_i) v(s_i) \notin \Omega \Big\}. \tag{IV.24}$$

Solving (IV.23) for the velocity component of the flow, we get

$$\begin{cases} v(s) = e^{-s} v_0 & \text{for } s \in [0, s_1), \\ v(s_i^+) = \mathcal{R}_{x(s_i)} v(s_i^-), & \\ v(s) = e^{-(s-s_i)} v(s_i^+) & \text{for } s \in (s_i, s_{i+1}), \end{cases} \tag{IV.25}$$

which gives the particle trajectory, for $s \in (s_i, s_{i+1})$,

$$
\begin{aligned}
x(s) &= x_0 + \int_0^s v(\tau)\mathrm{d}\tau = x_0 + \sum_{k=0}^{i-1} \int_{s_k}^{s_{k+1}} v(\tau)\mathrm{d}\tau + \int_{s_i}^{s} v(\tau)\mathrm{d}\tau \\
&= x_0 + \sum_{k=0}^{i-1} \left(1 - e^{-(s_{k+1}-s_k)}\right) v(s_k^+) + \left(1 - e^{-(s-s_k)}\right) v(s_i^+).
\end{aligned}
$$

Instead of considering an exponentially decreasing velocity $v(s)$ on an infinite interval $[0,\infty)$, we would like to consider particle trajectories with constant speed on a finite interval $[0,1)$. To that end, we notice that the reflection operator $\mathcal{R}$ is isometric, which means

$$
\begin{aligned}
v(s_i^+) &= \mathcal{R}_{x(s_i)}\big(v(s_i^-)\big) \\
&= \mathcal{R}_{x(s_i)}\big(e^{-(s_i-s_{i-1})}v(s_{i-1}^+)\big) \\
&= e^{-(s_i-s_{i-1})}\mathcal{R}_{x(s_i)} \circ \mathcal{R}_{x(s_{i-1})}\big(e^{-(s_{i-1}-s_{s-2})}v(s_{i-2}^+)\big) \\
&= e^{-(s_i-s_{i-2})}\mathcal{R}_{x(s_i)} \circ \mathcal{R}_{x(s_{i-1})} \circ \mathcal{R}_{x(s_{i-2})}\big(e^{-(s_{i-2}-s_{s-3})}v(s_{i-3}^+)\big) \\
&= e^{-(s_i-0)}\mathcal{R}_{x(s_i)} \circ \mathcal{R}_{x(s_{i-1})} \circ \cdots \circ \mathcal{R}_{x(s_1)}\big(v_0\big).
\end{aligned}
$$

We define the operator R^i as

$$
\begin{cases}
R^0 = Id, \\
R^i = \mathcal{R}_{x(s_i)} \circ R^{i-1},
\end{cases}
\tag{IV.26}
$$

and a new velocity $w(s) := e^s v(s)$ which then satisfies

$$
\begin{cases}
w(s) = v_0 & \text{for } s \in (0, s_1), \\
w(s_i) = R^i v_0, & \\
w(s) = R^i w(s_i) & \text{for } s \in [s_i, s_{i+1}).
\end{cases}
\tag{IV.27}
$$

It is easy to check that for any s, $|w(s)| = |v_0|$. The trajectory $x(s)$ can be written, with the new velocity variable w as

$$
\begin{aligned}
x(s) &= x_0 + \int_0^s e^{-\tau} w(\tau)\mathrm{d}\tau \\
&= x_0 + \sum_{k=0}^{i-1} \left(e^{-s_k} - e^{-s_{k+1}}\right) w(s_k) + \left(e^{-s} - e^{-s_i}\right) w(s_i)
\end{aligned}
$$

and finally, we introduce a new parametrisation $\tau = 1 - e^{-s} \in [0,1)$ and the corresponding reflection times $\tau_i = 1 - e^{-s_i}$ with which we have, for any $\tau \in [\tau_i, \tau_{i+1})$,

$$
\begin{cases}
x(\tau) = x_0 + \sum_{k=0}^{i-1} (\tau_{k+1} - \tau_k)\, w(\tau_k) + (\tau - \tau_i)\, w(\tau_i), \\
w(\tau) = w(\tau_i) = R^i w_0.
\end{cases}
\tag{IV.28}
$$

We notice that the particle trajectory $x(\tau)$ together with the velocity profile $w(\tau)$ in (IV.28) can be seen as the specular cycle associated with our Hamiltonian dynamics. Next, we record a couple of simple observations on the forward exit times s_i and the grazing set Σ_0 associated with the specular cycle (IV.28).

Lemma IV.4.1. *Let $\Omega \subset \mathbb{R}^d$ be strictly convex. Then, we have*

(i) For any $(x, v) \in \Omega \times \mathbb{R}^d$, the trajectory never passes through a grazing set Σ_0.

(ii) For any $(x, v) \in \Omega \times \mathbb{R}^d$, there exists a $N \in \mathbb{N}^*$ depending on (x, v) such that the forward exit time $s_{N+1}(x, v)$ does not exist.

The above result is proved in [SV97, Chapter 1, Section 1.3, Lemma 1.3.17], an excellent book of Safarov and Vassiliev, where geodesic billiards on manifolds are extensively studied.

IV.4.2 Solution to the auxiliary problem and rescaling

Next, we shall define a function on the phase space.

Definition IV.4.1 (End-point function)**.** *The end-point function $\eta : \left(\bar{\Omega}\times\mathbb{R}^d\right)\backslash\Sigma_0 \to \bar{\Omega}$ is defined such that for every $(x_0, v_0) \in \bar{\Omega} \times \mathbb{R}^d \setminus \Sigma_0$,*

$$
\eta(x_0, v_0) = x(\tau = 1),
$$

where the particle trajectory is given in (IV.28).

Using the end-point function $\eta(x,v)$, we have a solution to the auxiliary problem (IV.21) for any

$$\psi \in \mathfrak{D} := \left\{\psi \in C^\infty(\overline{\Omega}) \text{ such that } \nabla\psi \cdot n(x) = 0 \text{ for } x \in \partial\Omega\right\}, \tag{IV.29}$$

which can be explicitly written as

$$\varphi(x,v) = \psi\left(\eta(x,v)\right).$$

Hence we deduce a solution to the auxiliary problem (IV.22) for any $\psi \in \mathfrak{D}$ and for any $\varepsilon > 0$,

$$\phi^\varepsilon(x,v) = \psi\left(\eta(x,\varepsilon v)\right). \tag{IV.30}$$

Indeed, the end-point function ensures not only that ϕ^ε is constant along the specular cycles, which in turns implies that the first two equations of (IV.22) are satisfied, but also that $\phi^\varepsilon(x,0) = \psi\big(\eta(x,0)\big) = \psi(x)$.
For ϕ^ε to be a test function in the weak formulation (IV.11) of Vlasov-Fokker-Planck equation, we need to add a dependency in time. Hence taking $\psi(t,x) \in \mathfrak{D}$ for all $t \in [0,T]$, we have

$$\phi^\varepsilon(t,x,v) = \psi(t,\eta(x,\varepsilon v)).$$

Finally, to conclude this section about the auxiliary problem, let us determine the limit of the family $\phi^\varepsilon(t,x,v)$ as ε goes to 0. By the definition of $\eta(x,v)$, for any $(x,v) \in \Omega \times \mathbb{R}^d$, there exists ε small enough, namely $\varepsilon < \mathrm{dist}(x,\partial\Omega)/|v|$, such that $\eta(x,\varepsilon v) = x + \varepsilon v$. Therefore

$$\lim_{\varepsilon\to 0} \psi\left(t,\eta(x,\varepsilon v)\right) = \lim_{\varepsilon\to 0} \psi\left(t, x+\varepsilon v\right) = \psi(t,x) \quad \forall (t,x,v) \in [0,T]\times\Omega\times\mathbb{R}^d.$$

IV.5 Derivation of the macroscopic model

We now return to the proof of Theorem IV.1.1. Consider $f^\varepsilon(t,x,v)$ a weak solution of (IV.5a)-(IV.5b)-(IV.5c) in the sense of Definition IV.3.1. For any ϕ^ε satisfying (IV.10) we have

$$\iiint\limits_{(0,T)\times\Omega\times\mathbb{R}^d} f^\varepsilon(t,x,v)\Big(\varepsilon^2\partial_t\phi^\varepsilon + \varepsilon v\cdot\nabla_x\phi^\varepsilon - v\cdot\nabla_v\phi^\varepsilon + \Delta_v\phi^\varepsilon\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ + \varepsilon^2\iint\limits_{\Omega\times\mathbb{R}^d} f^{in}(x,v)\phi^\varepsilon(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0. \tag{IV.31}$$

In particular, for $\phi^\varepsilon(t,x,v) = \psi\left(t,\eta(x,\varepsilon v)\right)$, where $\psi(t,x)\in\mathfrak{D}\;\forall t\in[0,T]$, we have

$$\begin{cases} \varepsilon v\cdot\nabla_x\left[\psi\left(t,\eta(x,\varepsilon v)\right)\right] - v\cdot\nabla_v\left[\psi\left(t,\eta(x,\varepsilon v)\right)\right] = 0 \\ \Delta_v\left[\psi\left(t,\eta(x,\varepsilon v)\right)\right] = \varepsilon^2\Delta_v\left[\psi\left(t,\eta(x,\cdot)\right)\right](\varepsilon v). \end{cases}$$

Hence, (IV.31) becomes

$$\iiint\limits_{(0,T)\times\Omega\times\mathbb{R}^d} f^\varepsilon\left(\partial_t\psi + \Delta_v\left[\psi\left(t,\eta(x,\cdot)\right)\right](\varepsilon v)\right)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \tag{IV.32}$$

$$+\iint\limits_{\Omega\times\mathbb{R}^d} f^{in}(x,v)\psi\left(0,\eta(x,\varepsilon v)\right)\,\mathrm{d}v\,\mathrm{d}x = 0. \tag{IV.33}$$

Since f^ε converges weakly* in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathcal{M}^{-1}(v)\mathrm{d}x\mathrm{d}v))$ (Proposition IV.3.2), in order to take the limit as ε goes to 0 we need to show that $\Delta_v\left[\psi\left(t,\eta(x,\cdot)\right)\right](\varepsilon v)$ converges strongly in $\mathrm{L}^2(\mathcal{M}(v)\mathrm{d}x\mathrm{d}v)$. To that end, we write

$$\begin{aligned}
\Delta_v&\left[\psi\left(t,\eta(x,\cdot)\right)\right](\varepsilon v) = \nabla_v\cdot\nabla_v\left(\psi\left(t,\eta(x,\cdot)\right)\right)(\varepsilon v) \\
&= \sum_{i=1}^d\sum_{k=1}^d \frac{\partial^2\eta_k}{\partial v_i^2}(x,\varepsilon v)\frac{\partial\psi}{\partial\eta_k}(t,\eta(x,\varepsilon v)) \\
&\quad + \varepsilon^2\sum_{i=1}^d\sum_{k=1}^d\sum_{l=1}^d \frac{\partial\eta_k}{\partial v_i}(x,\varepsilon v)\frac{\partial^2\psi}{\partial\eta_k\partial\eta_l}(t,\eta(x,\varepsilon v))\frac{\partial\eta_l}{\partial v_i}(x,\varepsilon v) \\
&= \Delta_v\eta(x,\varepsilon v)\cdot\nabla_x\psi\left(t,\eta(x,\varepsilon v)\right) + \mathrm{Tr}\left(\nabla_v\eta(x,\varepsilon v)^\top\nabla_v\eta(x,\varepsilon v)H_x\psi\left(t,\eta(x,\varepsilon v)\right)\right),
\end{aligned} \tag{IV.34}$$

where $H_x\psi$ denotes the Hessian matrix of ψ. For any $(x,v)\in\Omega\times\mathbb{R}^d$ we know that for ε small enough, i.e. $\varepsilon < \mathrm{dist}(x,\partial\Omega)/|v|$, $\eta(x,\varepsilon v) = x+\varepsilon v$, which means $\nabla_v\eta(x,\varepsilon v) = \mathrm{Id}$ and $\Delta_v\eta(x,\varepsilon v) = 0$ so that, using the computation above, for such ε we have

$$\Delta_v\left[\psi\left(t,\eta(x,\cdot)\right)\right](\varepsilon v) = \mathrm{Tr}\left(H_x\psi\left(t,x+\varepsilon v\right)\right) = \Delta_x\psi\left(t,x+\varepsilon v\right).$$

Since ψ is smooth, this yields a point-wise convergence

$$\Delta_v \left[\psi\left(t, \eta(x,\cdot)\right)\right](\varepsilon v) \to \Delta_x \psi(t,x) \quad \text{a.e. on } [0,T]\times\Omega. \tag{IV.35}$$

This convergence holds up to the boundary, indeed for any $x \in \partial\Omega$ and $v \in \mathbb{R}^d$ we see, by the definition of the end-point function, that for some ε small enough

$$\nabla_v \eta(x, \varepsilon v) = \begin{cases} \mathrm{Id} & \text{if } v\cdot n(x) < 0 \\ \mathrm{Id} - 2n(x)\otimes n(x) & \text{if } v\cdot n(x) > 0 \end{cases}$$

which yields, in turn, that $\Delta\eta(x,\varepsilon v) = 2n(x)\delta_{v\cdot n(x)=0}$. Hence, for any $\psi \in \mathfrak{D}$ and $(x,v) \in \partial\Omega\times\mathbb{R}^d$ we have

$$\Delta_v \left[\psi\left(t, \eta(x,\cdot)\right)\right](\varepsilon v) \to 2\nabla\psi(x)\cdot n(x)\delta_{v\cdot n(x)=0} + \Delta\psi(x) = \Delta\psi(x).$$

Finally, we have the following.

Lemma IV.5.1. *If Ω is a unit ball in $\mathbb{R}^d$ and η is defined as in Definition IV.4.1 on Ω then we have*

$$\sup_{r>0}\Big(\Delta_v \left[\psi\left(t,\eta(x,\cdot)\right)\right](rv)\Big) \in \mathrm{L}^\infty\big((0,T);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})\big) \tag{IV.36}$$

for any $\psi \in \mathfrak{D}_T$, where

$$\mathfrak{D}_T := \big\{\psi \in \mathcal{C}^\infty([0,T)\times\overline{\Omega}) \text{ s.t. } \psi(T,\cdot) = 0 \text{ and } n(x)\cdot\nabla_x\psi(t,x) = 0 \text{ on } (0,T)\times\partial\Omega\big\}.$$

To prove this lemma we study the regularity of the end-point function $\eta(x,v)$, which is rather technical and will be the subject of Appendix A. Nevertheless, this allows us to use the Lebesgue's dominated convergence theorem in $\mathrm{L}^2(\mathcal{M}(v)\mathrm{d}x\mathrm{d}v)$ and pass to the limit in the weak formulation (IV.32) as ε goes to 0 to get

$$\iint\limits_{(0,T)\times\Omega} \rho(t,x)\Big(\partial_t\psi(t,x) + \Delta_x\psi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t + \int\limits_\Omega \rho^{in}(x)\psi(0,x)\,\mathrm{d}x = 0, \tag{IV.37}$$

which holds for any $\psi \in \mathfrak{D}_T$. To conclude the proof of Theorem IV.1.1, we need to show that the solution ρ of (IV.37) is a weak solution to the diffusion equation (IV.7a)-(IV.7b)-(IV.7c), which is the objective of the following proposition.

Proposition IV.5.2. *If ρ satisfies, for every $\psi \in \mathfrak{D}_T$,*

$$\iint\limits_{(0,T)\times\Omega} \rho(t,x)\Big(\partial_t\psi + \Delta_x\psi\Big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega} \rho^{in}(x)\psi(0,x)\,\mathrm{d}x = 0, \tag{IV.38}$$

then ρ is the unique solution of the heat equation with homogeneous Neumann boundary condition, i.e., for any $\psi \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)$,

$$\int\limits_0^T \langle \partial_t\rho,\, \psi\rangle_{V',V}\,\mathrm{d}t + \iint\limits_{(0,T)\times\Omega} \nabla_x\rho(t,x)\cdot\nabla_x\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{IV.39}$$

where $V = \mathrm{H}^1(\Omega)$ and V' is its topological dual.

Proof. This proof consists in showing that the solution ρ of (IV.38) is regular enough for (IV.39) to make sense. Once this is established, a classical density argument will conclude the proof of the proposition, and therefore the proof of Theorem IV.1.1, by showing that (IV.39) holds for any ψ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega)\big)$.

For any $u \in C^\infty([0,T);C_c^\infty(\Omega))$, we consider the unique solution to the boundary-value problem

$$\begin{cases} \Delta_x\psi(t,x) = \dfrac{\partial u}{\partial x_i}(t,x) & \text{in } (0,T)\times\Omega, \\ \nabla\psi(t,x)\cdot n(x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ \displaystyle\int\limits_\Omega \psi(t,x)\,\mathrm{d}x = 0, & \end{cases} \tag{IV.40}$$

for any $i \in \{1,\cdots,d\}$. Notice that the time variable t in (IV.40) plays the role of a parameter. It is well known that the solution ψ to (IV.40) will be in $\mathfrak{D}_T$. To derive the energy estimate, multiply (IV.40) by ψ and integrate over Ω yielding

$$\int\limits_\Omega \psi(t,x)\Delta_x\psi(t,x)\,\mathrm{d}x = \int\limits_\Omega \psi(t,x)\frac{\partial u}{\partial x_i}(t,x)\,\mathrm{d}x \quad \forall t\in[0,T]. \tag{IV.41}$$

On the left-hand side, the homogeneous Neumann condition in (IV.40) yields

$$\left|\int\limits_\Omega \psi(t,x)\Delta_x\psi(t,x)\,\mathrm{d}x\right| = \|\nabla\psi(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)} \quad \forall t\in[0,T].$$

On the right hand-side of (IV.41), since u is compactly supported in Ω we can write

$$\begin{aligned}\left|\int_\Omega \psi(t,x)\frac{\partial u}{\partial x_i}(t,x)\,\mathrm{d}x\right| &= \left|\int_\Omega u(t,x)\frac{\partial \psi}{\partial x_i}(t,x)\,\mathrm{d}x\right| \\ &\leq \|u(t,\cdot)\|_{\mathrm{L}^2(\Omega)}\|\nabla\psi(t,\cdot)\|_{\mathrm{L}^2(\Omega)} \quad \forall t\in[0,T].\end{aligned}$$

Together with the Poincaré inequality, this computation shows that $\|\psi(t,\cdot)\|_{\mathrm{L}^2(\Omega)} \leq \|u(t,\cdot)\|_{\mathrm{L}^2(\Omega)}$ for all $t\in[0,T]$. Taking the thus constructed $\psi(t,x)$ as the test function in the formulation (IV.38), we get

$$\left|\iint_{(0,T)\times\Omega} \rho(t,x)\frac{\partial u}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}t\right| \leq \left|\int_\Omega \rho^{in}(x)\psi(0,x)\,\mathrm{d}x\right| + \left|\iint_{(0,T)\times\Omega} \rho(t,x)\partial_t\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right|,$$

which, in particular, implies that for any $u\in\mathcal{D}(\Omega)$, considering ψ that doesn't depend on t and with a constant $C=\|\rho^{in}\|_{\mathrm{L}^2(\Omega)}$, we arrive at the following control

$$\left|\int_0^T \left\langle \frac{\partial\rho}{\partial x_i}, u\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\mathrm{d}t\right| \leq C\|u\|_{\mathrm{L}^2(\Omega)}.$$

The above observation implies that

$$\rho\in\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)).$$

It is a classical matter to show that $\mathfrak{D}_T$ is dense in $\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$. Using the above regularity of ρ in (IV.38) and taking $\psi\in\mathrm{L}^2(0,T;\mathrm{H}^1(\Omega))$ would yield the following regularity on the time derivative

$$\partial_t\rho\in\mathrm{L}^2(0,T;V'),$$

where V' is the topological dual of $V=\mathrm{H}^1(\Omega)$. Thus, we have proved that the limit local density $\rho(t,x)$ is the unique solution of the weak formulation (IV.39). □

Remark IV.5.3. *Note that the result in Lemma IV.5.1 is given for a particular choice of the spatial domain – a ball in $\mathbb{R}^d$. We are unable so far to prove a similar regularity result in more general strictly convex domains.*

Chapter V

Anomalous diffusion limit with diffusive boundary

Joint work with Antoine Mellet and Marjolaine Puel

Contents

V.1 Introduction

This chapter is based on an on-going project A. Mellet and M. Puel and it may be interpreted as a continuation of the work presented in Chapter III. Note however that we are not able, to this day, to give a full rigorous proof of the anomalous diffusion limit we investigate. As consequence, let us begin this chapter by a summary of the method we develop in order to state clearly what is rigorously proven and what is still formal.

First, we briefly present in section V.1.1 the fractional Vlasov-Fokker-Planck equation with diffusive boundary condition, rescale the equation appropriately for the anomalous diffusion limit and establish in section V.2.1 a priori estimates which entail convergence of the solution f_ε to the rescaled equation towards the kernel of the collision operator.

Second, we introduce in section V.2.2 the associated auxiliary problem in the spirit of the ones introduced for absorption and specular reflection boundary condition. We notice, however, that because of the diffusive boundary condition we will require some assumption on the initial condition $\psi(t,x)$ of the auxiliary problem in order to ensure existence and construct solutions. We define an extension of the function ψ to the complementary of the domain Ω thanks to which we are able to write explicitly the condition needed by the initial condition ψ_ε (which now depends on ε) in order to have a solution.

This leads to the main difficulty that we are still unable solve so far, namely the proper construction of the sequence ψ_ε of well-prepared initial conditions from any given function ψ in $\mathcal{D}([0,T)\times\bar{\Omega})$. Furthermore, although we can write explicitly a solution to the auxiliary problem from such initial data ψ_ε, its regularity is still undetermined hence we are not able to control its convergence as ε tends to 0. As a consequence, we only take formally the limit in the variational formulation of the kinetic equation, section V.2.3, in order to identify a limit non-local diffusion operator $\mathcal{L}$ and an associated non-local boundary operator $\mathcal{D}^{2s-1}$.

Finally, we study rigorously this limit operator $\mathcal{L}$ in section V.3, proving an integration by parts formula, defining an associated Hilbert space $\mathcal{H}^s_{\text{diff}}(\Omega)$ and deriving a Poincaré-type inequality on a sub-space of $\mathcal{H}^s_{\text{diff}}(\Omega)$ of functions with zero mean.

Note that well-posedness of the limit non-local diffusion equation is still not established. Although we defined the appropriated Hilbert space $\mathcal{H}^s_{\text{diff}}(\Omega)$ where one should look for weak solutions of this problem it is still unclear how to construct a subspace of smooth functions dense in $\mathcal{H}^s_{\text{diff}}(\Omega)$ which would allow to close a Lax-Milgram argument as explained in Remark V.3.3.

V.1.1 Kinetic equation with diffusive boundary condition

We investigate the long time/small mean-free-path asymptotic behaviour of the solution of the fractional Vlasov-Fokker-Planck (VFP) equation:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (vf) - (-\Delta_v)^s f \qquad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^d, \tag{V.1a}$$

$$f(0,x,v) = f_{in}(x,v) \qquad \text{in } \Omega \times \mathbb{R}^d, \tag{V.1b}$$

for $s \in (0,1)$ on a smooth convex domain Ω. We introduce the oriented set:

$$\Sigma_\pm = \{(x,v) \in \Sigma; \pm n(x) \cdot v > 0\} \text{ with } \Sigma = \partial\Omega \times \mathbb{R}^d \tag{V.2}$$

where $n(x)$ is the outgoing normal vector and we denote by γf the trace of f on $\mathbb{R}^+ \times \partial\Omega \times \mathbb{R}^d$. The boundary conditions then take the form of a balance between the values of the traces of f on these oriented sets $\gamma_\pm f := \mathbb{1}_{\Sigma_\pm} \gamma f$. We consider in the chapter the diffusive boundary condition:

$$\gamma_- f(t,x,v) = \mathcal{B}[\gamma_+ f](t,x,v) := c_0 F(v) \int_{\Sigma_+^x} \gamma_+ f(t,x,w) |w \cdot n(x)| \, \mathrm{d}w \tag{V.3}$$

with the normalising constant c_0 given by

$$c_0 = \left(\int_{v \cdot n(x) \leq 0} F(v) |v \cdot n(x)| \, \mathrm{d}v \right)^{-1}. \tag{V.4}$$

where F is the unique normalised equilibrium of the fractional Fokker-Planck operator:

$$\mathcal{L}^s(F) := \nabla_v \cdot (vF) - \big(-\Delta \big)^s F = 0, \qquad \int_{\mathbb{R}^d} F(v) \, \mathrm{d}v = 1. \tag{V.5}$$

Note that, although we have an explicit formula for the Fourier transform of F, we do not have an explicit F is physical variable but we know that it is heavy-tailed:

$$F(v) \sim \frac{C}{|v|^{d+2s}}, \qquad \text{as } |v| \to +\infty. \tag{V.6}$$

We refer to Chapter III and the introduction of this thesis for a detailed presentation of the fractional Vlasov-Fokker-Planck equation and the associated bibliography. The diffusive boundary condition (V.3) was introduced by Maxwell in [Max79] to model the diffusive properties of the boundary. When considered in a linear combination with the specular reflection condition (see Chapter III), it gives rise to the Maxwell boundary condition

$$\gamma_- f(t,x,v) = \alpha\gamma_+ f\big(t,x,\mathcal{R}_x(v)\big) + (1-\alpha)\mathcal{B}[\gamma_+ f](t,x,v)$$

for some $\alpha \in (0,1)$ and we refer to [Cer00] or [Mis10] for more information on this type of boundary condition.
In order to investigate the long time/small mean-free-path asymptotic behaviour of the solution of (V.1a)-(V.1b)-(V.3) we introduce the Knudsen number ε and the anomalous rescaling

$$t' = \varepsilon^{2s-1} t$$

and multiply the collision operator by $1/\varepsilon$ to model the number of collisions per unit of time going to infinity. The rescaled fractional VFP equation with diffusive boundary condition then reads

$$\varepsilon^{2s-1}\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon = \frac{1}{\varepsilon}\Big(\nabla_v\cdot(v f_\varepsilon) - \big(-\Delta_v\big)^s f_\varepsilon\Big) \qquad \text{in } [0,T)\times\Omega\times\mathbb{R}^d \quad \text{(V.7a)}$$

$$f_\varepsilon(0,x,v) = f_{in}(x,v) \qquad \text{in } \Omega\times\mathbb{R}^d \quad \text{(V.7b)}$$

$$\gamma_- f_\varepsilon(t,x,v) = \mathcal{B}[\gamma_+ f_\varepsilon](t,x,v) \qquad \text{on } [0,T)\times\Sigma_-. \quad \text{(V.7c)}$$

Although the existence of weak solutions to such kinetic equations has been established in similar situations, see [Car98], [MV07] or [ASC16], the question of regularity of such solutions, especially up to the boundary, is a challenging issue. It has been investigated, for instance, by S. Mischler in [Mis10]. Nevertheless, it is common when studying anomalous limits of kinetic equations, to resort to a definition of weak solutions that does not involve the trace of that solution in order to avoid such considerations.

Definition V.1.1. *We say that f_ε is a weak solution of* (V.7a)-(V.7b)-(V.7c) *on $Q_T = [0,T)\times\Omega\times\mathbb{R}^d$ if for all test function $\phi\in\mathcal{D}(Q_T)$ such that*

$$\gamma_+\phi(t,x,v) = \mathcal{B}^*[\gamma_-\phi](t,x,v) \qquad \forall (t,x,v)\in[0,T)\times\Sigma_+ \qquad \text{(V.8)}$$

the following equality holds:

$$\begin{aligned}\iiint\limits_{Q_T} f_\varepsilon\Big(\partial_t\phi + \varepsilon^{-2s}\big[\varepsilon v\cdot\nabla_x\phi - v\cdot\nabla_v\phi\big] - \varepsilon^{-2s}\big(-\Delta_v\big)^s\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \\ = \iint\limits_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\phi(0,x,v)\,\mathrm{d}x\,\mathrm{d}v.\end{aligned} \tag{V.9}$$

Note that the adjoint operator $\mathcal{B}^*$ is defined as

$$\mathcal{B}^*[\gamma_-\phi](t,x,v) = c_0\int\limits_{\Sigma_-^x}\gamma_-\phi(t,x,w)|w\cdot n(x)|F(w)\,\mathrm{d}w$$

and it is actually independent of $v\in\Sigma_+^x$.

V.2 Anomalous diffusion limit

In the spirit of [Mel10], [CMT12] and [Ces16], the method we present here to establish the anomalous diffusion limit of (V.7a)-(V.7b)-(V.7c) consists of three steps. First, we establish a priori estimates that we ensure the convergence of f_ε towards the kernel of the fractional Fokker-Planck operator. Then, we introduce an auxiliary problem through which we take advantage of the particular properties of the kinetic model. And finally, we identify the limit of f_ε by taking the limit in the weak formulation (V.9) with the test functions constructed by the auxiliary problem.

V.2.1 Apriori estimates

The a priori estimates we derive for (V.7a)-(V.7b)-(V.7c) are exactly the same as the ones we established in Chapter III when we considered the same equation with absorption or specular reflection on the boundary. The key ingredient is the dissipativity of the fractional Fokker-Planck operator:

Proposition V.2.1. *For all f smooth enough, if we define the dissipation as:*

$$\mathcal{D}^s(f) := -\int\limits_{\mathbb{R}^d}\mathcal{L}^s(f)\frac{f}{F}\,\mathrm{d}v \tag{V.10}$$

then there exists $\theta > 0$ such that

$$\mathcal{D}^s(f) = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\big(f(v)-f(w)\big)^2}{|v-w|^{d+2s}} \frac{\mathrm{d}vw}{F(v)} \geq \theta \int\limits_{\mathbb{R}^d} \big|f(v) - \rho F(v)\big|^2 \frac{\mathrm{d}v}{F(v)} \tag{V.11}$$

where $\rho = \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v$. Note, in particular, that $\mathcal{D}^s(f) \geq 0$.

We refer to Chapter III for the proof of this proposition. It allows us to prove the following:

Proposition V.2.2. *Let f_{in} be in $L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)$ and s be in $(0,1)$. The weak solution f^ε of the rescaled fractional Vlasov-Fokker-Planck equation* (V.7a)-(V.7b) *with diffusive boundary condition* (V.7c)*, converges when ε goes to 0 as follows*

$$f^\varepsilon(t,x,v) \rightharpoonup \rho(t,x)F(v) \text{ weakly in } L^\infty\big(0,T; L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)\big) \tag{V.12}$$

where $\rho(t,x)$ is the limit of the macroscopic densities $\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon\,\mathrm{d}v$.

Proof. We follow the same line of reasoning as Chapter III: assuming existence and uniqueness of a weak solution to (V.7a)-(V.7b)-(V.7c) satisfying appropriate estimates, we multiply (V.7a) by $f_\varepsilon/F(v)$ and integrate over x and v to get

$$\varepsilon^{2s-1}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega\times\mathbb{R}^d}(f_\varepsilon)^2\frac{\mathrm{d}x\,\mathrm{d}v}{F(v)} + \iint\limits_{\Sigma}\gamma f_\varepsilon^2 v\cdot n(x)\frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)} + \frac{1}{\varepsilon}\mathcal{D}^s(f_\varepsilon) = 0 \tag{V.13}$$

For the boundary term, we write

$$\begin{aligned}
&\iint\limits_{\Sigma}\gamma f_\varepsilon^2 v\cdot n(x)\frac{\mathrm{d}\sigma\,\mathrm{d}v}{F(v)}\\
&\quad= \iint\limits_{\Sigma_+}\gamma_+ f_\varepsilon^2 n(x)\cdot v\frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)} - \iint\limits_{\Sigma_-}\gamma_- f_\varepsilon^2 n(x)\cdot v\frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)}\\
&\quad= \iint\limits_{\Sigma_+}\gamma_+ f_\varepsilon^2 n(x)\cdot v\frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)} - \iint\limits_{\Sigma_-}\Bigg(c_0F(v)\int\limits_{\Sigma_+^x}\gamma_+ f_\varepsilon n(x)\cdot w\,\mathrm{d}w\Bigg)^2|v\cdot n(x)|\frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)}.
\end{aligned}$$

Using Cauchy-Schwartz' inequality on the second term on the right-hand-side we get:

$$\iint\limits_{\Sigma_-} \left(F(v) \int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon n(x)\cdot w \,\mathrm{d}w \right)^2 |v\cdot n(x)| \frac{\mathrm{d}\sigma(x)\,\mathrm{d}v}{F(v)}$$
$$\leq c_0^2 \int\limits_{\partial\Omega} \left(\int\limits_{\Sigma_+^x} \gamma_+ f_\varepsilon^2 |n(x)\cdot w| \frac{\mathrm{d}w}{F(w)} \right) \left(\int\limits_{\Sigma_+^x} |n(x)\cdot w| F(w)\,\mathrm{d}w \right) \left(\int\limits_{\Sigma_-^x} |n(x)\cdot v| F(v)\,\mathrm{d}v \right) \mathrm{d}\sigma(x)$$
$$\leq \iint\limits_{\Sigma_+} \gamma_+ f_\varepsilon^2 |n(x)\cdot w| \frac{\mathrm{d}w}{F(w)}$$

hence

$$\iint\limits_{\Sigma} \gamma f_\varepsilon^2 v\cdot n(x) \frac{\mathrm{d}\sigma\,\mathrm{d}v}{F(v)} \geq 0.$$

As a consequence, (V.13) gives us a uniform bound on f_ε in $L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d)$. Moreover we know that the dissipation controls the distance between f_ε and the kernel of the fractional Fokker-Planck operator. Hence, since the uniform bound of f_ε implies the boundedness of ρ_ε, this yields the weak convergence of f_ε to $\rho F(v)$ in $L^\infty(0,T;L^2_{F^{-1}(v)}(\Omega\times\mathbb{R}^d))$ where ρ is the weak limit of ρ_ε. □

V.2.2 Auxiliary problem

In the spirit of the method introduced in [CMT12] and [Ces16], we want to introduce an auxiliary problem through which we build a particular sub-class of test functions that will allows to take the limit in the weak formulation and establish the anomalous diffusion limit. The natural auxiliary problem associated with (V.9) reads for $\psi \in \mathcal{D}([0,T)\times\bar{\Omega})$:

$$\varepsilon v\cdot\nabla_x\phi_\varepsilon - v\cdot\nabla_v\phi_\varepsilon = 0 \qquad \text{in } [0,T)\times\Omega\times\mathbb{R}^d \tag{V.14a}$$
$$\phi_\varepsilon(t,x,0) = \psi(t,x) \qquad \text{in } [0,T)\times\Omega \tag{V.14b}$$
$$\gamma_+\phi_\varepsilon(t,x,v) = \mathcal{B}^*[\gamma_-\phi_\varepsilon](t,x) \qquad \text{on } [0,T)\times\Sigma_+. \tag{V.14c}$$

However, unlike absorption or specular reflection boundary conditions, the diffusive boundary condition (V.7c) is non-local in velocity. As a consequence, its adjoint form (V.14b) will interact with the transport-like problem (V.14a)-(V.14b) and induce the need for extra assumptions on the initial condition ψ in order for the auxiliary problem to be well-posed. More precisely, we can construct a particular solution of the auxiliary problem as follows:

Proposition V.2.3. *Consider $\psi \in \mathcal{D}([0,T)\times\bar\Omega)$ and define its extension $\widetilde{\psi} : [0,T)\times \mathbb{R}^d\times\mathbb{R}^d \mapsto \mathbb{R}$ which coincides with ψ on $\bar\Omega$ in the sense that*

$$\widetilde{\psi}(t,x,v) = \psi(t,x) \qquad \text{in } [0,T)\times\bar\Omega\times\mathbb{R}^d \tag{V.15}$$

and is defined, for $x\in\mathbb{R}^d\setminus\Omega$ as the solution of:

$$v\cdot\nabla_x\widetilde{\psi}(t,x,v) = 0 \qquad \text{in } [0,T)\times\big(\mathbb{R}^d\setminus\Omega\big)\times\mathbb{R}^d \tag{V.16a}$$

$$\widetilde{\psi}(t,x,v) = \psi(t,x) \qquad \text{in } [0,T)\times\Sigma_+. \tag{V.16b}$$

Moreover, define the operator $\mathcal{D}_\varepsilon^{2s-1}$ as

$$\mathcal{D}_\varepsilon^{2s-1}[\psi](t,x) = \varepsilon^{1-2s}\int_{\mathbb{R}^d}\big[\widetilde{\psi}(t,x+\varepsilon v,v)-\psi(t,x)\big]vF(v)\,\mathrm{d}v. \tag{V.17}$$

Then, for any $\psi_\varepsilon\in\mathcal{D}([0,T)\times\bar\Omega)$ such that

$$\mathcal{D}_\varepsilon^{2s-1}[\psi_\varepsilon](t,x)\cdot n(x) = 0 \qquad \text{on } [0,T)\times\partial\Omega \tag{V.18}$$

the function ϕ_ε given by

$$\phi_\varepsilon(t,x,v) = \widetilde{\psi}_\varepsilon(t,x+\varepsilon v,v)$$

is a solution of the auxiliary problem (V.14a)-(V.14b)-(V.14c).

Before we prove this proposition, let us gives some details on what the extension $\widetilde{\psi}$ entails. First, note that the set Σ_+ of outgoing velocities for $\Omega\times\mathbb{R}^d$ is the set of incoming velocities of $(\mathbb{R}^d\setminus\Omega)\times\mathbb{R}^d$ so the boundary-value-problem (V.16a)-(V.16b) makes sense in an Analysis of PDE point of view.
Second, one can integrate the equation along lines $(x(s),v(x))$ satisfying $\dot x = v$ and $\dot v = 0$ to get a formula for $\widetilde{\psi}$. Namely, as a consequence of the convexity of Ω, the value of ψ at x on the boundary is propagated outside Ω along these lines $(x+sv,v)$ with $v\cdot n(x)\geq 0$ and $s>0$:

$$\widetilde{\psi}(t,x+sv,v) = \psi(t,x). \tag{V.19}$$

However, for $x \notin \Omega$, there are lines $(x+sv, v)$ with $v \in \mathbb{R}^d$ and $s > 0$ that do not intersect the domain Ω which means that the problem (V.15)-(V.16a)-(V.16b) does not have a unique solution. This will not be of great importance in our analysis because we will systematically consider $x \in \Omega$ but we can set $\widetilde{\psi}$ to be 0 along those lines, which amounts formally to taking a homogeneous Dirichlet condition at infinity.
Finally, since the boundary condition (V.16b) does not depend on v, note that the function $s \mapsto \widetilde{\psi}(t, x, sv)$ is constant for any $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$ hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\widetilde{\psi}(t,x,sv)\right]\Big|_{s=1} = v \cdot \nabla_v \widetilde{\psi}(t,x,v) = 0 \qquad \text{for all } x \in \mathbb{R}^d \text{ and } v \in \mathbb{R}^d. \tag{V.20}$$

Proof of Proposition V.2.3. Given the expression of ϕ_ε, (V.14b) is immediate and it is easy to check that (V.14a) is satisfied:

$$\begin{aligned}
\varepsilon v \cdot \nabla_x \phi_\varepsilon - v \cdot \nabla_v \phi_\varepsilon &= \varepsilon v \cdot \nabla_x \widetilde{\psi}_\varepsilon - v \cdot \left[\varepsilon \nabla_x \widetilde{\psi}_\varepsilon + \nabla_v \widetilde{\psi}_\varepsilon\right] \\
&= -v \cdot \nabla_v \widetilde{\psi}_\varepsilon \\
&= 0
\end{aligned}$$

using (V.20). Furthermore, for the boundary condition (V.14c), we see on the one hand that thanks to (V.19) for all $(x, v) \in \Sigma_+$

$$\gamma_+ \phi_\varepsilon(t,x,v) = \widetilde{\psi}_\varepsilon(t, x+\varepsilon v, v) = \psi(t,x)$$

and on the other hand, we have

$$\mathcal{B}^*[\gamma_- \phi_\varepsilon](t,x) = c_0 \int_{\Sigma_-^x} \widetilde{\psi}_\varepsilon(t, x+\varepsilon w, w)|w \cdot n(x)|F(w)\,\mathrm{d}w$$

so that the boundary condition (V.14c) actually reads

$$\psi(t,x) = c_0 \int_{\Sigma_-^x} \widetilde{\psi}_\varepsilon(t, x+\varepsilon w, w)|w \cdot n(x)|F(w)\,\mathrm{d}w.$$

Since c_0 is the normalising constant (V.4) we can place $\psi(t,x)$ under the integral and multiply the equality by ε^{1-2s} to recover

$$\varepsilon^{1-2s} \int_{\Sigma_-^x} \left[\widetilde{\psi}_\varepsilon(t, x+\varepsilon w, w) - \psi(t,x)\right]|w \cdot n(x)|F(w)\,\mathrm{d}w = 0.$$

Finally, noticing that for $v \in \Sigma_+^x$ the relation (V.19) ensures that the integrand is null, we can actually integrate over $v \in \mathbb{R}^d$ and recover (V.18) which concludes the proof. □

This auxiliary problem differs significantly from the ones we introduced in the absorption or the specular reflection case because it requires assumption on the test function ψ. In order to close our method, we would need to be able, from any ψ in a well-chosen sub-space of $\mathcal{D}([0,T) \times \bar{\Omega})$, to define a sequence ψ_ε such that for any $\varepsilon > 0$, ψ_ε satisfies (V.18) and ψ_ε converges to ψ in a sense that needs to be defined carefully. The construction of this sequence ψ_ε is a non-trivial problem, in particular because of the non-local nature of the boundary condition (V.18) and we are still not sure how to do it. However, unlike the specular reflection case where the function ϕ_ε was much less regular that ψ, here we see that the regularity of ψ, or more precisely the regularity of its extension $\widetilde{\psi}$, transfers immediately to ϕ_ε.

V.2.3 Formal asymptotics

Since we have not been able, so far, to construct the test functions ψ_ε appropriately, we cannot derive the macroscopic equation on ρ rigorously from the rescaled kinetic equation. However, we can formally identify the non-local diffusion operator that should arise in the limit.

The weak formulation (V.9) with test function ϕ_ε solution of the auxiliary problem reads

$$\iiint_{Q_T} f_\varepsilon\big(\partial_t \phi_\varepsilon - \varepsilon^{-2s}\big(-\Delta_v\big)^s \phi_\varepsilon\big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v = \iint_{\Omega\times\mathbb{R}^d} f_{in}(x,v)\phi_\varepsilon(0,x,v)\,\mathrm{d}x\,\mathrm{d}v \tag{V.21}$$

In the spirit of [Mel10], and keeping in mind that we proved in Section V.2.1 the convergence of f_ε towards $\rho(t,x)F(v)$, we introduce the operator $\mathcal{L}^\varepsilon$ defined as

$$\mathcal{L}^\varepsilon[\psi](t,x) = \varepsilon^{-2s}\int_{\mathbb{R}^d}\big(-\Delta_v\big)^s\big[\widetilde{\psi}_\varepsilon(t,x+\varepsilon v,v)\big]F(v)\,\mathrm{d}v. \tag{V.22}$$

This operator is directly related to the operator $\mathcal{D}_\varepsilon^{2s-1}$ defined in (V.17) as follows:

Proposition V.2.4. *For all function $\psi \in \mathcal{D}(\bar{\Omega})$ we have*

$$\mathcal{L}^\varepsilon[\psi](x) = -\nabla_x \cdot \mathcal{D}_\varepsilon^{2s-1}[\psi](x). \tag{V.23}$$

Proof. Using the fact that F is the equilibrium of the fractional Fokker-Planck operator, an integration by parts in the definition of $\mathcal{L}^\varepsilon$ yields

$$\begin{aligned}\mathcal{L}^\varepsilon[\psi](x) &= \varepsilon^{-2s}\int_{\mathbb{R}^d}\widetilde{\psi}(x+\varepsilon v, v)\big(-\Delta_v\big)^s F(v)\,\mathrm{d}v\\ &= \varepsilon^{-2s}\int_{\mathbb{R}^d}\widetilde{\psi}(x+\varepsilon v, v)\nabla_v\cdot(vF)\,\mathrm{d}v\\ &= -\varepsilon^{-2s}\int_{\mathbb{R}^d}\Big[\varepsilon v\cdot\nabla_x\widetilde{\psi}(x+\varepsilon v,v) + v\cdot\nabla_v\widetilde{\psi}(x+\varepsilon v,v)\Big]F(v)\,\mathrm{d}v.\end{aligned}$$

Further, using (V.20) and the fact that $\int_{\mathbb{R}^d} vF(v)\,\mathrm{d}v = 0$, we deduce

$$\begin{aligned}\mathcal{L}^\varepsilon[\psi](x) &= -\varepsilon^{1-2s}\nabla_x\cdot\int_{\mathbb{R}^d}\widetilde{\psi}(x+\varepsilon v, v)vF(v)\,\mathrm{d}v\\ &= -\varepsilon^{1-2s}\nabla_x\cdot\int_{\mathbb{R}^d}\big[\widetilde{\psi}(x+\varepsilon v,v)-\psi(t,x)\big]vF(v)\,\mathrm{d}v\\ &= -\nabla_x\cdot\mathcal{D}_\varepsilon^{2s-1}[\psi](x).\end{aligned}$$

□

We now investigate formally the limit of the operators $\mathcal{L}^\varepsilon$ and $\mathcal{D}_\varepsilon^{2s-1}$ as ε goes to 0. For $\mathcal{L}^\varepsilon$, with the integral definition of the fractional Laplacian, the change of variable $w=\varepsilon z$ and the fact that $\widetilde{\psi}(x,tw)=\widetilde{\psi}(x,w)$ for all $t>0$ we have

$$\begin{aligned}\mathcal{L}^\varepsilon[\psi](x) &= \varepsilon^{-2s}c_{d,s}P.V.\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\widetilde{\psi}(x+\varepsilon v,\varepsilon v)-\widetilde{\psi}(x+\varepsilon z,\varepsilon z)}{|v-z|^{d+2s}}F(v)\,\mathrm{d}w\,\mathrm{d}v\\ &= c_{d,s}P.V.\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\widetilde{\psi}(x+\varepsilon v,v)-\widetilde{\psi}(x+w,w)}{|\varepsilon v-w|^{d+2s}}F(v)\,\mathrm{d}v\,\mathrm{d}w,\end{aligned}$$

hence the formal limit

$$\mathcal{L}^\varepsilon[\psi](x)\underset{\varepsilon\to0}{\longrightarrow}\mathcal{L}[\psi](x) := c_{d,s}P.V.\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\psi(x)-\widetilde{\psi}(x+w,w)}{|w|^{d+2s}}F(v)\,\mathrm{d}w\,\mathrm{d}v, \qquad \text{(V.24)}$$

which can be written, after the change of variable $y = x + w$ and using the fact that F is normalised, as

$$\mathcal{L}[\psi](x) = c_{d,s} P.V. \int_{\mathbb{R}^d} \frac{\psi(x) - \widetilde{\psi}(y, y-x)}{|y-x|^{d+2s}} \, \mathrm{d}y. \tag{V.25}$$

Furthermore, we use the change of variable $y = x + \varepsilon v$ in the definition (V.17) of $\mathcal{D}_\varepsilon^{2s-1}$ to write

$$\mathcal{D}_\varepsilon^{2s-1}[\psi](x) = \int_{\mathbb{R}^d} \big[\widetilde{\psi}(y, y-x) - \psi(x)\big] \frac{y-x}{\varepsilon^{d+2s}} F\Big(\frac{y-x}{\varepsilon}\Big) \, \mathrm{d}y$$

and since $F(z/\varepsilon) \sim \varepsilon^{d+2s}/|z|^{d+2s}$ for small ε, we have formally

$$\mathcal{D}_\varepsilon^{2s-1}[\psi](x) \underset{\varepsilon \to 0}{\longrightarrow} \mathcal{D}^{2s-1}[\psi](x) := c_{d,s} P.V. \int_{\mathbb{R}^d} \big[\widetilde{\psi}(y, y-x) - \psi(x)\big] \frac{y-x}{|y-x|^{d+2s}} \, \mathrm{d}y. \tag{V.26}$$

Finally, passing to the limit in (V.23) we get

$$\mathcal{L}[\psi](x) = -\nabla_x \cdot \mathcal{D}^{2s-1}[\psi]. \tag{V.27}$$

Assuming all the necessary convergence hold, we would obtain in the limit of the weak formulation (V.21) the following non-diffusion equation and the associated notion of weak solution:

Definition V.2.1. *We say that ρ is a weak solution of the non-local diffusion equation*

$$\partial_t \rho + \mathcal{L}[\rho] = 0 \qquad \textit{in } [0,T) \times \Omega \tag{V.28a}$$

$$\rho(0,x) = \rho_{in}(x) \qquad \textit{in } \Omega \tag{V.28b}$$

$$\mathcal{D}^{2s-1}[\rho](t,x) \cdot n(x) = 0 \qquad \textit{on } [0,T) \times \partial\Omega \tag{V.28c}$$

if, for all test function $\psi \in \mathcal{D}([0,T) \times \bar{\Omega})$ satisfying (V.28c) *we have*

$$\iint_{[0,T)\times\Omega} \rho(t,x)\Big(\partial_t \psi(t,x) - \mathcal{L}[\psi](t,x)\Big) \, \mathrm{d}t \, \mathrm{d}x = \int_{\Omega} \rho_{in}(x) \psi(0,x) \, \mathrm{d}x \tag{V.29}$$

V.3 Analysis of the non-local operator

This section is devoted to the analysis of the non-local operator $\mathcal{L}$ defined in (V.25). Our purpose is to give some intuition as to what the non-local diffusion problem (V.28a)-(V.28b)-(V.28c) models and also to sketch some of the differences between this operator and the specular diffusion operator $(-\Delta)^s_{\mathrm{SR}}$. Indeed, one of our primary motivation in this chapter is to highlight one of the most crucial difference between classical and anomalous diffusion limits in bounded domain which is the fact that the limit equation that identify the particle density ρ is not the same, in the anomalous case, if we consider specular reflections on the boundary or the diffusive boundary conditions. This also highlights the pertinence of our method to derive non-diffusion equations in bounded domain from kinetic equations and its ability to define non-local operators, physically relevant by construction, and that seem new to the best of our knowledge. different from any operators previously defined such as those we have introduced in the introduction of this thesis.
We focus on three crucial results concerning the operator $\mathcal{L}$. First, an integration by parts formula which justifies Definition V.2.1 of a weak solution to (V.28a)-(V.28b)-(V.28c). Second, the construction of a Hilbert norm associated with this operator, in the same way the fractional Laplacian is related to the fractional Sobolev space, or the specular diffusion operator $(-\Delta)^s_{\mathrm{SR}}$ is related to $\mathcal{H}^s_{\mathrm{SR}}(\Omega)$. Third, a Poincaré-type inequality on a sub-space of the Hilbert space we construct, generalising the classical Poincaré inequality of $H^1_0(\Omega)$.

V.3.1 Integration by parts formula

The integration by parts formula we derive for $\mathcal{L}$ rests upon its relation with $\mathcal{D}^{2s-1}$:

Proposition V.3.1. *For any smooth functions ϕ and ψ*

$$\int_\Omega \psi\mathcal{L}[\phi]\,\mathrm{d}x - \int_\Omega \phi\mathcal{L}[\psi]\,\mathrm{d}x = -\int_{\partial\Omega}\Big(\psi\mathcal{D}^{2s-1}[\phi]\cdot n(x) - \phi\mathcal{D}^{2s-1}[\psi]\cdot n(x)\Big)\,\mathrm{d}\sigma(x). \tag{V.30}$$

Note that this expression is a natural generalisation of the integration by parts formula for the (non-fractional) Laplacian :

$$\int_\Omega \psi(-\Delta_x)\phi\,\mathrm{d}x - \int_\Omega \phi(-\Delta_x)\psi\,\mathrm{d}x = -\int_{\partial\Omega}\Big(\psi\nabla_x\phi\cdot n(x) - \phi\nabla_x\psi\cdot n(x)\Big)\,\mathrm{d}\sigma(x).$$

so (V.30) justifies the fact that (V.29) is the weak formulation of (V.28a)-(V.28b)-(V.28c).

Proof. Using (V.27), we have

$$\int_{\Omega} \psi \mathcal{L}[\phi] \, \mathrm{d}x = \int_{\Omega} \nabla \psi \cdot \mathcal{D}^{2s-1}[\phi] \, \mathrm{d}x - \int_{\partial\Omega} \psi \mathcal{D}^{2s-1}[\phi] \cdot n(x) \, \mathrm{d}\sigma(x).$$

The rest of the proof rests upon the following lemma:

Lemma V.3.2. *For all ϕ and ψ smooth enough*

$$\int_{\Omega} \mathcal{D}^{2s-1}[\phi] \cdot \nabla \psi \, \mathrm{d}x = \int_{\Omega} \mathcal{D}^{2s-1}[\psi] \cdot \nabla \phi \, \mathrm{d}x. \tag{V.31}$$

Postponing the proof of the lemma, we see that it entails

$$\begin{aligned}
\int_{\Omega} \psi \mathcal{L}[\phi] \, \mathrm{d}x &= \int_{\Omega} \nabla \phi \cdot \mathcal{D}^{2s-1}[\psi] \, \mathrm{d}x - \int_{\partial\Omega} \psi \mathcal{D}^{2s-1}[\phi] \cdot n(x) \, \mathrm{d}\sigma(x) \\
&= \int_{\Omega} \phi \mathcal{L}[\psi] \, \mathrm{d}x + \int_{\partial\Omega} \phi \mathcal{D}^{2s-1}[\psi] \cdot n(x) \, \mathrm{d}\sigma(x) - \int_{\partial\Omega} \psi \mathcal{D}^{2s-1}[\phi] \cdot n(x) \, \mathrm{d}\sigma(x)
\end{aligned}$$

where we recognise is the integration by parts formula (V.30). □

Proof of Lemma V.3.2. In order to prove (V.31), we will show that

$$\int_{\Omega} \mathcal{D}_{\varepsilon}^{2s-1}[\phi] \cdot \nabla_x \psi \, \mathrm{d}x = \int_{\Omega} \mathcal{D}_{\varepsilon}^{2s-1}[\psi] \cdot \nabla_x \phi \, \mathrm{d}x \tag{V.32}$$

and the result then follows by passing to the limit in ε. From the definition (V.17) of $\mathcal{D}_{\varepsilon}^{2s-1}$ where we notice that $\int vF(v) \, \mathrm{d}v = 0$, we write

$$\begin{aligned}
\int_{\Omega} \mathcal{D}_{\varepsilon}^{2s-1}[\phi] \cdot \nabla_x \psi \, \mathrm{d}x &= \varepsilon^{1-2s} \iint_{\Omega \times \mathbb{R}^d} \widetilde{\phi}(x+\varepsilon v, v) v \cdot \nabla_x \psi(x) F(v) \, \mathrm{d}v \, \mathrm{d}x \\
&= \varepsilon^{1-2s} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \widetilde{\phi}(x+\varepsilon v, v) v \cdot \nabla_x \widetilde{\psi}(x, -v) F(v) \, \mathrm{d}v \, \mathrm{d}x
\end{aligned}$$

where we used the definition of the extension $\widetilde{\psi}$, and more precisely (V.16a). With an integration by parts and a change of variable $y = x + \varepsilon v$ this yields

$$\int_\Omega \mathcal{D}_\varepsilon^{2s-1}[\phi] \cdot \nabla_x \psi \,\mathrm{d}x = -\varepsilon^{1-2s} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \widetilde{\psi}(x, -v) v \cdot \nabla_x \widetilde{\phi}(x + \varepsilon v, v) F(v) \,\mathrm{d}v \,\mathrm{d}x$$
$$= -\varepsilon^{1-2s} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \widetilde{\psi}(y - \varepsilon v, -v) v \cdot \nabla_x \widetilde{\phi}(y, v) \,\mathrm{d}v \,\mathrm{d}y.$$

Finally, using (V.16a) again and the change of variable $w = -v$ this yields

$$\int_\Omega \mathcal{D}_\varepsilon^{2s-1}[\phi] \cdot \nabla_x \psi = \varepsilon^{1-2s} \iint_{\Omega \times \mathbb{R}^d} \widetilde{\psi}(y + \varepsilon w, w) w \cdot \nabla_x \phi(y) F(w) \,\mathrm{d}w \,\mathrm{d}y$$
$$= \int_\Omega \mathcal{D}_\varepsilon^{2s-1}[\psi] \cdot \nabla_x \phi \,\mathrm{d}x$$

which is (V.32). □

V.3.2 The Hilbert space $\mathcal{H}^s_{\mathrm{diff}}(\Omega)$

As a consequence of the integration by parts formula (V.30) we see that if ϕ and ψ functions satisfying the boundary condition (V.28c) then we have

$$\int_\Omega \psi \mathcal{L}[\phi] \,\mathrm{d}x = \int_\Omega \phi \mathcal{L}[\psi] \,\mathrm{d}x \tag{V.33}$$

and we would like to see if we can deduce a semi-norm from $\mathcal{L}$. To that end, use the divergence form (V.27) and Lemma V.3.2 to write

$$\int_\Omega \psi \mathcal{L}[\phi] \,\mathrm{d}x + \int_\Omega \phi \mathcal{L}[\psi] \,\mathrm{d}x = \int_\Omega \Big(\nabla_x \psi \cdot \mathcal{D}^{2s-1}[\phi] + \nabla_x \phi \cdot \mathcal{D}^{2s-1} \psi \Big) \,\mathrm{d}x$$
$$= c_{d,s} P.V. \iint_{\Omega \times \mathbb{R}^d} \Big(\big[\widetilde{\phi}(y, y - x) - \phi(x)\big] (y - x) \cdot \nabla_x \psi(x)$$
$$+ \big[\widetilde{\psi}(y, y - x) - \psi(x)\big] (y - x) \cdot \nabla_x \phi(x) \Big) \frac{\mathrm{d}x \,\mathrm{d}y}{|x - y|^{d+2s}}.$$

Moreover, we know that for any $x \in \Omega$ and $y \in \mathbb{R}^d$, $(y-x) \cdot \nabla_x \widetilde{\psi}(y, y-x) = 0$ because on the one hand if $y \in \Omega$ then $\widetilde{\psi}(y, y - x) = \psi(y)$ does not depend on x, and on the other hand if $y \notin \Omega$ then with $v = y - x$ we have $(y - x) \cdot \nabla_x \widetilde{\psi}(y, y - x) = -v \cdot \nabla_v \widetilde{\psi}(y, v) = 0$

thanks to (V.20). Hence

$$\int_\Omega \psi \mathcal{L}[\phi]\,\mathrm{d}x + \int_\Omega \phi \mathcal{L}[\psi]\,\mathrm{d}x$$
$$= c_{d,s} P.V. \iint_{\Omega\times\mathbb{R}^d} \Big(\big[\widetilde{\phi}(y,y-x) - \phi(x)\big](y-x)\cdot\nabla_x\big[\psi(x) - \widetilde{\psi}(y,y-x)\big]$$
$$+ \big[\widetilde{\psi}(y,y-x) - \psi(x)\big](y-x)\cdot\nabla_x\big[\phi(x) - \widetilde{\phi}(y,y-x)\big]\Big) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+2s}}$$
$$= -c_{d,s} P.V. \iint_{\Omega\times\mathbb{R}^d} (y-x)\nabla_x \Big(\big[\psi(x) - \widetilde{\psi}(y,y-x)\big]\big[\phi(x) - \widetilde{\phi}(y,y-x)\big]\Big) \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{d+2s}}.$$

Integrating by parts, this yields

$$\int_\Omega \psi \mathcal{L}[\phi]\,\mathrm{d}x + \int_\Omega \phi \mathcal{L}[\psi]\,\mathrm{d}x$$
$$= c_{d,s} P.V. \iint_{\Omega\times\mathbb{R}^d} \big[\psi(x) - \widetilde{\psi}(y,y-x)\big]\big[\phi(x) - \widetilde{\phi}(y,y-x)\big]\nabla_x\cdot\left(\frac{y-x}{|y-x|^{d+2s}}\right)\mathrm{d}x\,\mathrm{d}y$$
$$- c_{d,s} P.V. \iint_{\Sigma} \big[\psi(x) - \widetilde{\psi}(y,y-x)\big]\big[\phi(x) - \widetilde{\phi}(y,y-x)\big]\frac{(y-x)\cdot n(x)}{|y-x|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}\sigma(x).$$

Here, we can compute the divergence of $(y-x)/|y-x|^{d+2s}$ and notice that the integral over Σ_+ in the second term is null thanks to (V.19) which gives us, using (V.33)

$$\int_\Omega \psi\mathcal{L}\phi\,\mathrm{d}x = \frac{2sc_{d,s}}{2} P.V. \iint_{\Omega\times\mathbb{R}^d} \big[\psi(x) - \widetilde{\psi}(y,y-x)\big]\big[\phi(x) - \widetilde{\phi}(y,y-x)\big]\frac{1}{|y-x|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$+ \frac{c_{d,s}}{2} P.V. \iint_{\Sigma_-} \big[\psi(x) - \widetilde{\psi}(y,y-x)\big]\big[\phi(x) - \widetilde{\phi}(y,y-x)\big]\frac{\big|(y-x)\cdot n(x)\big|}{|y-x|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}\sigma(x). \tag{V.34}$$

As a consequence, we see now that we can indeed define a semi-norm, which we denote $[\cdot]_{\mathcal{H}^s_{\text{diff}}(\Omega)}$ as:

$$[\psi]^2_{\mathcal{H}^s_{\text{diff}}(\Omega)} := \tilde{c}_{d,s} P.V. \iint\limits_{\Omega\times\mathbb{R}^d} \frac{\big(\psi(x) - \widetilde{\psi}(y, y-x)\big)^2}{|y-x|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y \tag{V.35}$$

$$+ \frac{c_{d,s}}{2} P.V. \iint\limits_{\Sigma_-} \big(\psi(x) - \widetilde{\psi}(y, y-x)\big)^2 \frac{\big|(y-x)\cdot n(x)\big|}{|y-x|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}\sigma(x) \tag{V.36}$$

and the associated Hilbert space $\mathcal{H}^s_{\text{diff}}(\Omega)$ can then be defined as

$$\mathcal{H}^s_{\text{diff}}(\Omega) = \Big\{\psi \in L^2(\Omega) :\ [\psi]_{\mathcal{H}^s_{\text{diff}}(\Omega)} < \infty\Big\}. \tag{V.37}$$

Note that $[\cdot]_{\mathcal{H}^s_{\text{diff}}(\Omega)}$ is indeed a semi-norm, in the spirit of the Gagliardo semi-norm for the fractional Sobolev spaces, since any constant function cancels it. This is why the norm we use on $\mathcal{H}^s_{\text{diff}}(\Omega)$ is

$$\|\psi\|^2_{\mathcal{H}^s_{\text{diff}}(\Omega)} := \|\psi\|^2_{L^2(\Omega)} + [\psi]^2_{\mathcal{H}^s_{\text{diff}}(\Omega)} \tag{V.38}$$

and the scalar product is the sum of the scalar product of $L^2(\Omega)$ and (V.34).

Remark V.3.3. *The space $\mathcal{X}$ defined as*

$$\mathcal{X} = \Big\{\psi \in L^2(0,T;\mathcal{H}^s_{\text{diff}}(\Omega)) : \mathcal{D}^{2s-1}[\psi](x)\cdot n(x) = 0 \text{ on } \partial\Omega\Big\}$$

is the natural setting to find weak solutions of (V.28a)-(V.28b)-(V.28c) *in the sense of Definition V.2.1. To establish well-posedness of such diffusion equations, a classical line of reasoning, see for instance [Car98] and the previous chapters of this thesis, consists in considering an associated equation formally derived from* (V.28a) *for $u(t,x) = e^{-\lambda t}\rho(t,x)$ for some $\lambda > 0$, which reads*

$$\partial_t u + \lambda u + \mathcal{L}[u] = 0.$$

with initial condition and boundary condition resulting from (V.28b) *and* (V.28c). *Indeed, existence of solution for this problem in $\mathcal{X}$ is equivalent to existence for* (V.28a)-(V.28b)-(V.28c). *Moreover, this associated problem has the right structure for a Lax-*

Milgram argument: we can define the bilinear form

$$a(u,\varphi) = \iint\limits_{[0,T)\times\Omega} \Big(- u\partial_t\varphi + \lambda u\varphi + u\mathcal{L}[\varphi]\Big)\,\mathrm{d}t\,\mathrm{d}x$$

and the continuous bounded linear form

$$L(\varphi) = \int\limits_{\Omega} \rho_{in}(x)\varphi(0,x)\,\mathrm{d}x.$$

However, in order to close this Lax-Milgram argument and prove existence of solution in the sense of Definition V.2.1 we would need a dense subspace of $\mathcal{X}$ of smooth functions (such as the test functions ϕ in the weak formulation) and defining such a functional space is still an open problem.

V.3.3 A Poincaré-type inequality for $\mathcal{L}$

We introduce the space $\mathcal{H}^s_{\mathrm{diff},0}(\Omega)$ defined by

$$\mathcal{H}^s_{\mathrm{diff},0}(\Omega) := \Big\{ u \in \mathcal{H}^s_{\mathrm{diff}}(\Omega) : \int\limits_{\Omega} u\,\mathrm{d}x = 0 \Big\} \tag{V.39}$$

and note that on this space the semi-norm $[\cdot]_{\mathcal{H}^s_{\mathrm{diff}}(\Omega)}$ is actually a norm. We want to prove the following Poincaré-type inequality on $\mathcal{H}^s_{\mathrm{diff},0}(\Omega)$:

Lemma V.3.4. *If Ω is a smooth bounded open set of $\mathbb{R}^d$ then there exists a constant $C = C(d,s,\Omega)$ such that for all $\psi \in \mathcal{H}^s_{\mathrm{diff},0}(\Omega)$:*

$$\|\psi\|_{L^2(\Omega)} \leq C[\psi]_{\mathcal{H}^s_{\mathrm{diff},0}(\Omega)} \tag{V.40}$$

where $[\cdot]_{\mathcal{H}^s_{\mathrm{diff},0}(\Omega)}$ is the norm induced on the subspace $\mathcal{H}^s_{\mathrm{diff},0}(\Omega)$ of $\mathcal{H}^s_{\mathrm{diff}}(\Omega)$ by the seminorm $[\cdot]_{\mathcal{H}^s_{\mathrm{diff}}(\Omega)}$

Proof of Lemma V.3.4. This proof will use results from [DPV12] by E. Di Nezza, G. Palatucci and E. Valdinoci. Namely, from [DPV12, Theorem 5.4] we know that if we consider the Gagliardo seminorm on $H^s(\Omega)$ given by

$$[\psi]^2_{H^s(\Omega)} := c_{d,s} P.V. \iint\limits_{\Omega\times\Omega} \frac{(\psi(x)-\psi(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

on the smooth bounded domain Ω, then $H^s(\Omega)$ is continuously embedded in $H^s(\mathbb{R}^d)$ i.e. we can define an extension $\bar{\psi}$ of ψ to $\mathbb{R}^d$ such that $\bar{\psi}(x)\big|_\Omega = \psi(x)$ and

$$\|\bar{\psi}\|_{H^s(\mathbb{R}^d)} \leq C\|\psi\|_{H^s(\Omega)}. \tag{V.41}$$

In their paper, they construct this extension explicitly (see [DPV12, Section 5, Lemmas 1, 2 and 3 and the proof of Theorem 5.4]) and it is compactly supported in $\mathbb{R}^d$ (although its support extends beyond Ω). Furthermore, [DPV12, Theorem 6.5] states that if ψ is measurable and compactly supported in $\mathbb{R}^d$ then there exists a constant $C = C(d, s)$ such that

$$\|\psi\|_{L^{2^\star}(\mathbb{R}^d)} \leq C[\psi]_{H^s(\mathbb{R}^d)} \tag{V.42}$$

where $2^\star$ is the "fractional critical exponent" given by $2^\star = 2d/(d-2s)$.
Now, adapting the line of reasoning from the paper [SV12] of R. Servadei and E. Valdinocci, we consider $\psi \in \mathcal{H}^s_{\text{diff},0}(\Omega)$ and write for some constant $C = C(d, s, \Omega)$ (which may change along the computation but remains independent of ψ)

$$\|\psi\|_{L^2(\Omega)} \leq C\|\psi\|_{L^{2^\star}(\Omega)} \leq C\|\bar{\psi}\|_{L^{2^\star}(\Omega)}$$

where $\bar{\psi}$ is the extension of ψ to $\mathbb{R}^d$ mentioned above. The first inequality holds because $2^\star \geq 2$ and $|\Omega| < \infty$, and the second inequality holds because $\bar{\psi}(x)\big|_\Omega = \psi(x)$. Then, from [DPV12, Theorem 6.5] since $\bar{\psi}$ is compactly supported and measurable

$$\|\bar{\psi}\|_{L^{2^\star}(\Omega)} \leq C\|\bar{\psi}\|_{L^{2^\star}(\mathbb{R}^d)} \leq C[\bar{\psi}]_{H^s(\mathbb{R}^d)} \leq C[\psi]_{H^s(\Omega)}$$

where the third inequality is a consequence of [DPV12, Theorem 5.4]. Finally, we conclude the proof of the Poincaré inequality by noticing from the definition of $[\cdot]_{\mathcal{H}^s_{\text{diff},0}(\Omega)}$ that

$$[\psi]_{H^s(\Omega)} \leq C[\psi]_{\mathcal{H}^s_{\text{diff},0}(\Omega)}$$

since

$$
\begin{aligned}
[\psi]^2_{\mathcal{H}^s_{\mathrm{diff},0}(\Omega)} =& s[\psi]^2_{H^s(\Omega)} + \tilde{c}_{d,s} \iint\limits_{\Omega\times(\mathbb{R}^d\setminus\Omega)} \frac{\big(\psi(x) - \widetilde{\psi}(y, y-x)\big)^2}{|y-x|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&+ \frac{c_{d,s}}{2} \iint\limits_{\Sigma_-} \big(\psi(x) - \widetilde{\psi}(y, y-x)\big)^2 \frac{\big|(y-x)\cdot n(x)\big|}{|y-x|^{d+2s}} \,\mathrm{d}y\,\mathrm{d}\sigma(x)
\end{aligned}
$$

where the extra terms are positive. □

Appendix A

Free transport equation in a sphere

Contents

In this appendix, we call Ω the unit ball in $\mathbb{R}^d$ and we consider the trajectories in Ω described by (IV.28) and the associated η function. We recall that what we name "*trajectory that starts from $x \in \Omega$ with velocity $v \in \mathbb{R}^d$*" the trajectory that consists of straight lines, specularly reflected upon hitting the boundary, and that stops when the length of the trajectory is $|v|$, as illustrated by Figure III.2 in Section III.4.
We first note that a trajectory in Ω is necessarily included in a plane of dimension 2. Indeed, by definition of the specular reflection, when the trajectory hits the boundary, the reflected velocity is a linear combination of the initial velocity and the normal vector: for $t \in [0, 1]$ such that $|x + tv| = 1$, $Rv = v - 2(n(x + tv) \cdot v)n(x + tv)$ where is $n(x + tv) = x + tv$ because Ω is the unit ball. Since the normal vector belongs to the plane generated by x and v we see that the reflected velocity also belongs to that same plane, and every reflected velocities along this trajectory. As a consequence, we restrict the study of the regularity of η in a ball to the case of a disk in dimension $d = 2$.

A.0.1 Explicit expression of the trajectories

Consider (x, v) in $\Omega \times \mathbb{R}^2$, we call k the number of reflections that the trajectory which starts at x with velocity v undergoes. We also introduce

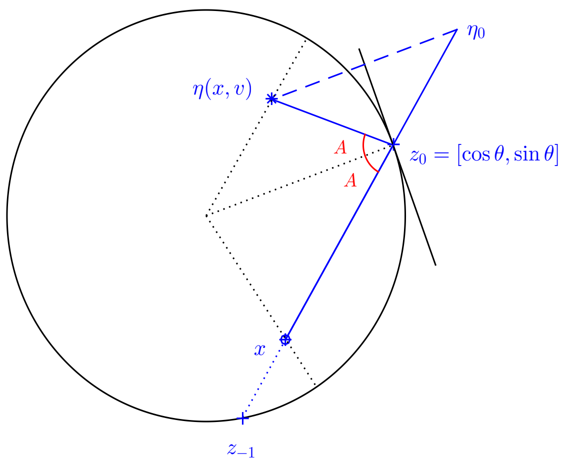
- θ such that $[\cos\theta, \sin\theta]$ is the first point of reflection,
- A the angle between the vector v and the outward normal to $\partial\Omega$ at $[\cos\theta, \sin\theta]$ (which, in the unit ball, is $[\cos\theta, \sin\theta]$ itself),
- $z_j = \big[\cos\big(\theta + j(\pi - 2A)\big), \sin\big(\theta + j(\pi - 2A)\big)\big]$ for any $j \in \mathbb{Z}$. Note that z_0 is the first point of reflection.

Proposition A.0.1. *For any $k \geq 0$ we have*

$$\eta(x, v) = k\big(z_{k-1} - z_k\big) + R_{k(\pi-2A)}(x + v) \tag{A.1}$$

where $R_{k(\pi-2A)}$ is the matrix of the rotation of angle $k(\pi - 2A)$.

Fig. A.1 Trajectory with 1 reflection in the circle



Proof. We will prove the expressions (A.1) by induction on the number of reflections. When $k = 0$, by definition of η we have $\eta(x, v) = x + v$ so that (A.1) holds.
Let us assume (IV.28) holds for some $k \geq 0$. Then, if we write $\eta_k = k(z_{k-1} - z_k) + R_{k(\pi-2A)}(x + v)$, we can compute $\eta(x, v)$ after $k + 1$ reflections using the relation

$$\eta(x, v) - z_k = R_{\pi-2A}(\eta_k - z_k)$$

as illustrated in Figure A.1 in the case $k = 0$. By definition of z_j we notice that $R_{\pi-2A}\, z_j = z_{j+1}$ hence:

$$\begin{aligned}
\eta(x, v) &= z_k + R_{\pi-2A}\Big(k(z_{k-1} - z_k) + R_{k(\pi-2A)}(x + v) - z_k\Big) \\
&= z_k + k(z_k - z_{k+1}) + R_{(k+1)(\pi-2A)}(x + v) - z_{k+1} \\
&= (k + 1)(z_k - z_{k+1}) + R_{(k+1)(\pi-2A)}(x + v)
\end{aligned}$$

which is exactly (A.1). □

A.0.2 First and second derivatives

We recall that $\mathfrak{D}_T$ is defined in Chapter III as:

$$\mathfrak{D}_T(\Omega) = \Big\{\psi \in \mathcal{C}^\infty([0, T) \times \bar{\Omega}) \text{ s.t. } \psi(T, \cdot) = 0 \text{ and } \forall x \in \partial\Omega : \nabla_x\psi(t, x) \cdot n(x) = 0\Big\}.$$

This section is devoted to the proof of the following estimates on the Jacobian matrix and the second derivative of η:

Lemma A.0.2. *Consider the unit ball Ω. The associated function η, defined in Section III.4.1, satisfies*

$$\|\nabla_v\eta(x, v)\| \in L^\infty(\Omega \times \mathbb{R}^d) \tag{A.2}$$

and for all ψ is in $\mathfrak{D}_T$

$$\Big\| D_v^2\Big[\psi\big(\eta(x, v)\big)\Big]\Big\| \in L^p_{F(v)}(\Omega \times \mathbb{R}^d). \tag{A.3}$$

for $p < 3$ where $\|\cdot\|$ is a matrix norm. Moreover,

$$\sup_{v\in\mathbb{R}^d}\Big\| D_v^2\Big[\psi\big(\eta(x, v)\big)\Big]\Big\| \in L^{2-\delta}(\Omega). \tag{A.4}$$

for any $\delta > 0$.

Proof. When $k = 0$, we have immediately $\nabla_v \eta = Id$ and controls (A.2) and (A.3) follow. When $k \geq 1$ we notice that for all j, $z_j = R_{k(\pi-2A)}[z_{j-k}]$ so that we have

$$\eta(x, v) = R_{k(\pi-2A)}\big(x + v - k(z_0 - z_{-1})\big)$$

where z_0 and z_{-1} are illustrated in Figure A.1. Also, we introduce the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which is the equivalent of the multiplication by i in complex coordinates – note that it commutes with the rotation matrix $R_{k(\pi-2A)}$ – and with which the Jacobian matrix of η with respect to v takes the form

$$\begin{aligned}
\nabla_v \eta(x, v) &= \Big[S R_{k(\pi-2A)}\big(x + v - k(z_0 - z_{-1})\big)\Big] \\
&\qquad \otimes \big(k \nabla_v (\pi - 2A)\big) + R_{k(\pi-2A)} \nabla_v \Big(x + v - k(z_0 - z_{-1})\Big) \\
&= \Big[S R_{k(\pi-2A)}\big(x + v - k(z_0 - z_{-1})\big)\Big] \\
&\qquad \otimes \big(- 2k \nabla_v A\big) - k R_{k(\pi-2A)} \nabla_v \big(z_0 - z_{-1}\big) + R_{k(\pi-2A)}.
\end{aligned}$$

Now, to differentiate the angles θ and A with respect to $v = (v_1, v_2)$, let us recall for t such that $|x + tv| = 1$ we have

$$\begin{cases} x_1 + t v_1 = \cos\theta \\ x_2 + t v_2 = \sin\theta \end{cases}$$

so that $v_2(\cos\theta - x_1) = v_1(\sin\theta - x_2)$, hence:

$$\frac{\partial \theta}{\partial v_1} = \frac{x_2 - \sin\theta}{v_1 \cos\theta + v_2 \sin\theta} = \frac{-t v_2}{|v| \cos A}, \qquad \frac{\partial \theta}{\partial v_2} = \frac{\cos\theta - x_1}{v_1 \cos\theta + v_2 \sin\theta} = \frac{t v_1}{|v| \cos A}.$$

Moreover, t satisfies $|v| \cos A = (x + tv) \cdot v = x \cdot v + t|v|^2$ which means

$$\frac{\partial \theta}{\partial v_1} = \frac{-v_2}{|v|} \frac{1}{|v| \cos A} \left(\cos A - \frac{x \cdot v}{|v|} \right), \qquad \frac{\partial \theta}{\partial v_2} = \frac{v_1}{|v|} \frac{1}{|v| \cos A} \left(\cos A - \frac{x \cdot v}{|v|} \right). \tag{A.5}$$

Also, by definition of A we have: $|v| \sin A = (x + tv) \times v = x_1 v_2 - x_2 v_1$ therefore:

$$\begin{aligned}
\frac{\partial A}{\partial v_1} &= \frac{-v_1(x_1 v_2 - x_2 v_1) - x_2(v_1^2 + v_2^2)}{|v|^3 \cos A}, & \frac{\partial A}{\partial v_2} &= \frac{-v_2(x_1 v_2 - x_2 v_1) + x_1(v_1^2 + v_2^2)}{|v|^3 \cos A} \\
&= \frac{-v_2}{|v|} \frac{1}{|v| \cos A} \left(\frac{x \cdot v}{|v|} \right) & &= \frac{v_1}{|v|} \frac{1}{|v| \cos A} \left(\frac{x \cdot v}{|v|} \right).
\end{aligned} \tag{A.6}$$

We now introduce the notations l_{in}, L and l_{end} defined as follows and illustrated in Figure A.2

- l_{in} is the distance between x and the first point of reflection z_0:

$$l_{in} = t|v| = \cos A - \frac{x \cdot v}{|v|}.$$

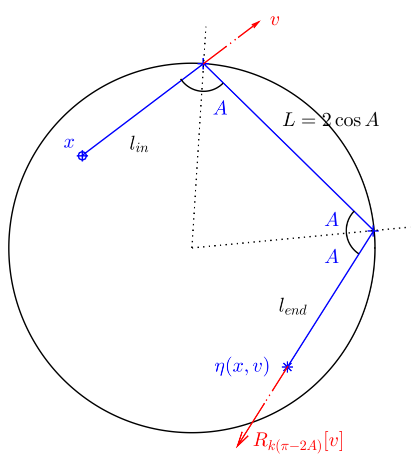
- L is the length between two consecutive reflections (note that it is constant because Ω is a ball):

$$L = 2\cos A$$

- l_{end} is the length between the last point of reflection and the end of the trajectory, $\eta(x, v)$:

$$l_{end} = |v| - (k-1)L - l_{in}.$$

Fig. A.2 Notations l_{in}, L and l_{end}



With these notations, the gradients of θ and A read

$$\nabla_v \theta = \left(\frac{2l_{in}}{|v|L}\right) S\frac{v}{|v|}, \qquad \nabla_v A = \left(\frac{L-2l_{in}}{|v|L}\right) S\frac{v}{|v|} \tag{A.7}$$

hence the Jacobian matrices of z_0 and z_1 as functions of v are

$$\begin{aligned}
\nabla_v z_0 &= Sz_0 \otimes \nabla_v \theta = \left(\frac{2l_{in}}{|v|L}\right) Sz_0 \otimes S\frac{v}{|v|},\\
\nabla_v z_{-1} &= Sz_{-1} \otimes \nabla_v\big(\theta - (\pi - 2A)\big) = \left(\frac{2(L-l_{in})}{|v|L}\right) Sz_{-1} \otimes S\frac{v}{|v|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\nabla_v \eta(x,v) &= SR_{k(\pi-2A)}\left[\left(\frac{2k(2l_{in}-L)}{|v|L}\right)\big(x+v-k(z_0-z_{-1})\big)\right] \otimes S\frac{v}{|v|}\\
&\quad - kSR_{k(\pi-2A)}\left[\left(\frac{2l_{in}}{|v|L}\right)z_0 - \left(\frac{2(L-l_{in})}{|v|L}\right)z_{-1}\right] \otimes S\frac{v}{|v|} + R_{k(\pi-2A)}\\
&= \frac{2k}{|v|L} SR_{k(\pi-2A)}\Big[(2l_{in}-L)\big(x+v-k(z_0-z_{-1})\big) - (l_{in} - \frac{L}{2})(z_0+z_{-1})\\
&\quad - \frac{L}{2}(z_0 - z_{-1})\Big] \otimes S\frac{v}{|v|} + R_{k(\pi-2A)}\\
&= \frac{2k}{|v|L} SR_{k(\pi-2A)}\Big[\frac{1}{2}(2l_{in}-L)(2x - z_0 - z_{-1}) + (2l_{in}-L)\big(v - k(z_0-z_{-1})\big)\\
&\quad - \frac{L}{2}(z_0 - z_{-1})\Big] \otimes S\frac{v}{|v|} + R_{k(\pi-2A)}.
\end{aligned}$$

Finally, by definition of z_0 and z_{-1} we see that

$$\begin{cases}
z_0 - z_{-1} = L\dfrac{v}{|v|},\\
x - z_0 = -l_{in}\dfrac{v}{|v|},\\
x - z_{-1} = (L - l_{in})\dfrac{v}{|v|}
\end{cases} \tag{A.8}$$

which yields

$$\nabla_v \eta(x,v) = \frac{2kL}{|v|}\left[2\frac{l_{in}}{L}\frac{l_{end}}{L} - \frac{l_{in}+l_{end}}{L}\right] SR_{k(\pi-2A)}\frac{v}{|v|} \otimes S\frac{v}{|v|} + R_{k(\pi-2A)}. \tag{A.9}$$

Introducing the notation

$$\underline{v} = \frac{v}{|v|}$$

as well as the angular function $\Theta : \mathbb{S}^1 \mapsto \mathcal{M}_2(\mathbb{R})$ and the function $\mu_x : \mathbb{R}^2 \mapsto \mathbb{R}$ as

$$\Theta(\underline{v}) = SR_{k(\pi-2A)}\underline{v} \otimes S\underline{v}. \tag{A.10}$$

$$\mu_x(v) = \frac{2kL}{|v|}\left[2\frac{l_{in}}{L}\frac{l_{end}}{L} - \frac{l_{in}+l_{end}}{L}\right]. \tag{A.11}$$

we have

$$\nabla_v \eta(x,v) = \mu_x(v)\Theta(\underline{v}) + R_{k(\pi-2A)}. \tag{A.12}$$

Now, since $|v| = l_{in} + (k-1)L + l_{end}$ we see that when $k > 1$:

$$\frac{kL}{|v|} = \frac{|v| + L - l_{in} - l_{end}}{|v|} \leq 1 + \frac{|L - l_{in} - l_{end}|}{|v|} \leq 2$$

and also, since $0 \leq l_{in}, l_{end} \leq L$ we have

$$-1 \leq 2\frac{l_{in}}{L}\frac{l_{end}}{L} - \frac{l_{in}+l_{end}}{L} \leq 0$$

so that

$$-4 \leq -2 - 2\frac{|L - l_{in} - l_{end}|}{|v|} \leq \mu_x(v) \leq 0. \tag{A.13}$$

Since $\|R_{k(\pi-2A)}\| = \|S\| = 1$, $\nabla_v \eta$ is bounded uniformly in x and v which concludes the proof of the control of $\nabla_v \eta$ stated in Proposition A.0.2. Notice that it also yields an explicit expression for the determinant:

$$\det \nabla_v \eta((x,v) = 1 + \frac{2kL}{|v|}\left[2\frac{l_{in}}{L}\frac{l_{end}}{L} - \frac{l_{in}+l_{end}}{L}\right] \tag{A.14}$$

from which is it easy to see that

$$-3 \leq -1 - 2\frac{|L - l_{in} - l_{end}|}{|v|} \leq \det \nabla_v \eta(x,v) \leq 1.$$

For the second derivative, we first notice that the expression of the Jacobian matrix above depends strongly on k and is not continuous when we go from k to $k+1$ which is equivalent to l_{end} going to 0 and l_{in} going to L. Hence, we introduce the sets E_k

defined as

$E_k = \{(x, v) \in \Omega \times \mathbb{R}^d$ s.t. the trajectory from (x, v) undergoes exactly k reflections$\}$

and the Jacobian of η actually reads

$$\nabla_v \eta(x, v) = \sum_{k\in\mathbb{N}} \nabla_v \eta^k(x, v) \mathbb{1}_{E_k}$$

where η^k is the expression (A.9). The second derivative of η will involve a derivative of the indicator functions of the E_k sets, i.e. the dirac measure of the boundary ∂E_k in the direction of the discontinuity. However, the boundary of E_k corresponds, by definition, to the (x, v) such that $\eta(x, v)$ is on $\partial\Omega$. Hence, similarly to the half-space case (see Section III.4.2.1) if we consider $\psi \in \mathfrak{D}_T$ then the direction of the jump will be orthogonal to $\nabla\psi$ at that point on $\partial\Omega$ and their product will be naught.
For the rest of this proof, we omit the dependence of η^k with respect to k. Before computing $D_v^2\eta$ which we define as usual as:

$$D_v^2\eta(x, v) = \begin{pmatrix} \partial_{11}^2\eta & \partial_{12}^2\eta \\ \partial_{21}^2\eta & \partial_{22}^2\eta \end{pmatrix} \tag{A.15}$$

where ∂_{ij}^2 means the second order partial derivative with respect to v_i and v_j, we feel it is simpler, given the form of the Jacobian matrix, to compute $\nabla_v \times \nabla_v\eta$ where we define the product $\times$ between a vector u in $\mathbb{R}^2$ and a matrix $M = (m_{ij})_{1\le i,j\le 2}$ in $\mathcal{M}_2(\mathbb{R})$ as

$$u \times M = \begin{pmatrix} m_{11}u & m_{12}u \\ m_{21}u & m_{22}u \end{pmatrix}$$

which means the product $u \times M$ is a vector valued matrix in $\mathcal{M}_2(\mathbb{R}^2)$. We write $\nabla_v\eta = (\partial_j\eta_i)_{i,j}$ and have

$$\nabla_v \times \nabla_v\eta = \begin{pmatrix} \nabla_v\partial_1\eta_1 & \nabla_v\partial_2\eta_1 \\ \nabla_v\partial_1\eta_2 & \nabla_v\partial_2\eta_2 \end{pmatrix}. \tag{A.16}$$

Using expression (A.9) we have:

$$\nabla_v \times \nabla_v\eta(x, v) = \nabla_v\mu_x(v) \times \Theta(\underline{v}) + \mu_x(v)\nabla_v \times \Theta(\underline{v}) + 2k\nabla_v A \times SR_{k(\pi-2A)} \tag{A.17}$$

Let us look at each of the terms individually and focus on singularities that might cause trouble for the integrability in $L^2_{F(v)}(\Omega \times \mathbb{R}^2)$, which in fact will arise when we get close to the grazing set, i.e. when L (as well as l_{in} and l_{end}) goes to 0 or, equivalently, when k goes to infinity. The simplest term to handle is the last one since we have, using (A.7):

$$2k\nabla_v A = \frac{1}{L}\left(\frac{2k\big(L - 2l_{in}\big)}{|v|}\right) S\underline{v} \tag{A.18}$$

so that

$$2k\nabla_v A \times SR_{k(\pi-2A)} := \frac{\alpha_A}{L} S\underline{v} \times SR_{k(\pi-2A)}$$

where α_A is uniformly bounded in x and v. For the second term,

$$\nabla_v \times \Theta(\underline{v}) = \nabla_v \times \Big(SR_{k(\pi-2A)}\underline{v} \otimes S\underline{v}\Big)$$

we introduce the extension of the dyadic product defined, for $u \in \mathbb{R}^2$ and $M \in \mathcal{M}_2(\mathbb{R})$ as:

$$u \otimes M = \begin{pmatrix} u_1 \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} & u_1 \begin{bmatrix} m_{21} \\ m_{22} \end{bmatrix} \\ u_2 \begin{bmatrix} m_{11} \\ m_{12} \end{bmatrix} & u_2 \begin{bmatrix} m_{21} \\ m_{22} \end{bmatrix} \end{pmatrix}$$

which is rather natural if one notices that for two vectors u and v, $u \otimes v = u\, v^T$, and we also define its commuted form $M \otimes u = (u \otimes M)^T$. With these notation, we have

$$\nabla_v \times \Theta(\underline{v}) = \big(\nabla_v SR_{k(\pi-2A)}\underline{v}\big) \otimes S\underline{v} + SR_{k(\pi-2A)}\underline{v} \otimes \big(\nabla_v S\underline{v}\big)$$

where on the one hand

$$\nabla_v S\underline{v} = \frac{-1}{|v|}\underline{v} \otimes S\underline{v}$$

and on the other hand

$$\begin{aligned}\nabla_v\big(SR_{k(\pi-2A)}\underline{v}\big) &= S\Big(SR_{k(\pi-2A)}\underline{v} \otimes \nabla_v\big(k(\pi - 2A)\big) + R_{k(\pi-2A)}\nabla_v \underline{v}\Big) \\ &= \frac{1}{L}\left(\frac{2k(L - 2l_{in})}{|v|} - \frac{L}{|v|}\right) R_{k(\pi-2A)}\underline{v} \otimes S\underline{v}.\end{aligned}$$

We get

$$\nabla_v \times \Theta(\underline{v}) = \frac{1}{L}\left(\frac{2k(L-2l_{in})}{|v|} - \frac{L}{|v|}\right)\left(R_{k(\pi-2A)}\underline{v} \otimes S\underline{v}\right) \otimes S\underline{v}$$
$$- \frac{1}{|v|} S R_{k(\pi-2A)}\underline{v} \otimes \left(\underline{v} \otimes S\underline{v}\right) \tag{A.19}$$

so that

$$\mu_x(v)\nabla_v \times \Theta(\underline{v}) = \frac{\alpha_\theta}{L}(R_{k(\pi-2A)}\underline{v} \otimes S\underline{v}) \otimes S\underline{v} + O(1)$$

when we are close to the grazing where, once again, α_θ is uniformly bounded in x and v. Note, in fact, that $\alpha_\theta = \alpha_A + O(L)$. Let us also note that the extension of the dyadic we defined is not quite associative in the sense that if u, v and w are vectors then

$$(u \otimes v) \otimes w = u \otimes (w \otimes v)$$

which we will keep in mind when we compute $D^2\eta(x, v)$. Finally, for the first term in (A.17) we notice that since $l_{in} = |x - z_0| = \sqrt{1 + |x|^2 - 2x \cdot z_0}$ we have

$$\nabla_v l_{in} = \frac{-2\nabla_v(x \cdot z_0)}{l_{in}} = \frac{-2x \cdot Sz_0}{l_{in}}\nabla_v\theta = \frac{-4x \cdot Sz_0}{|v|L} S\underline{v}$$

where $x \cdot Sz_0 = x \cdot S(x + tv) = t|v|x \cdot Sv/|v| = l_{in}\sin A$ so that in fact

$$\nabla_v l_{in} = \frac{-4l_{in}\sin A}{|v|L} S\underline{v}.$$

Moreover, $L = 2\cos A$ so we have

$$\nabla_v L = \frac{-2(L - 2l_{in})\sin A}{|v|L} S\underline{v}$$

and finally, $l_{end} = |v| - (k-1)L - l_{in}$ therefore

$$\nabla_v l_{end} = \underline{v} + \frac{1}{|v|L}\big(2(k-1)(L-2l_{in})\sin A + 4l_{in}\sin A\big)S\underline{v}$$
$$= \underline{v} + \frac{2\sin A}{L}\left(\frac{(k-1)(L-2l_{in})}{|v|} + \frac{2l_{in}}{|v|}\right)S\underline{v}.$$

Note that unlike $\nabla_v L$ and $\nabla_v l_{in}$, the gradient of l_{end} diverges in norm for small L (i.e. close to the grazing set) because the coefficient $\sin A/L$ goes to infinity. Differentiating $\mu_x(v)$ we get

$$\begin{aligned}
&\nabla_v\mu_x(v) = \nabla_v\Big(\frac{2kL}{|v|}\Big)\Big[2\frac{l_{in}}{L}\frac{l_{end}}{L} - \frac{l_{in}+l_{end}}{L}\Big]\\
&+\frac{2kL}{|v|}\Big[\nabla_v l_{end}\Big(2\frac{l_{in}}{L^2}-\frac{1}{L}\Big)+\nabla_v l_{in}\Big(2\frac{l_{end}}{L^2}-\frac{1}{L}\Big)+\nabla_v L\Big(\frac{l_{in}+l_{end}}{L^2}-4\frac{l_{in}l_{end}}{L^3}\Big)\Big]\\
&=\frac{-1}{|v|L}\mu_x(v)\Big(2\Big(1-2\frac{l_{in}}{L}\Big)S\underline{v}-L\underline{v}\Big)\\
&\quad+\frac{1}{L^2}\Big(\frac{2kL}{|v|}\Big)\Big(2\frac{l_{in}}{L}-1\Big)\Big[L\underline{v}+2\sin A\Big(\frac{(k-1)(L-2l_{in})}{|v|}+\frac{2l_{in}}{|v|}\Big)S\underline{v}\Big]\\
&\quad+\frac{1}{L}\frac{2kL}{|v|^2}\Big[\frac{4l_{in}\sin A}{L}\Big(1-2\frac{l_{end}}{L}\Big)-2\sin A\Big(1-2\frac{l_{in}}{L}\Big)\Big(\frac{l_{in}+l_{end}}{L}-4\frac{l_{in}l_{end}}{L^2}\Big)\Big]S\underline{v}.
\end{aligned} \tag{A.20}$$

Introducing uniformly bounded functions α_μ^i, $i\in\{1,2,3\}$ we get

$$\begin{aligned}
\nabla_v\mu_x(v)\times\Theta(\underline{v}) &= \frac{1}{L}\Big(\alpha_\mu^1 S\underline{v}+\alpha_\mu^2\underline{v}\Big)\times(SR_{k(\pi-2A)}\underline{v}\otimes S\underline{v})\\
&+\frac{1}{L^2}\alpha_\mu^3 S\underline{v}\times(SR_{k(\pi-2A)}\underline{v}\otimes S\underline{v}).
\end{aligned}$$

Together, all three terms yields

$$\begin{aligned}
\nabla_v\times\nabla_v\eta(x,v) &= \frac{1}{L}\Big(\alpha_\mu^1 S\underline{v}+\alpha_\mu^2\underline{v}\Big)\times(SR_{k(\pi-2A)}\underline{v}\otimes S\underline{v})\\
&+\frac{1}{L^2}\alpha_\mu^3 S\underline{v}\times(SR_{k(\pi-2A)}\underline{v}\otimes S\underline{v})\\
&+\frac{\alpha_\theta}{L}(R_{k(\pi-2A)}\underline{v}\otimes S\underline{v})\otimes S\underline{v}+\frac{\alpha_A}{L}S\underline{v}\times SR_{k(\pi-2A)}+O(1)
\end{aligned}$$

Identifying the terms in (A.15) with those of (A.16) we get

$$\begin{aligned}
D^2\eta(x,v) &= \frac{1}{L}SR_{k(\pi-2A)}\underline{v}\times\Big(\alpha_\mu^1(S\underline{v}\otimes S\underline{v})+\alpha_\mu^2(\underline{v}\otimes S\underline{v})+\frac{1}{L}\alpha_\mu^3(S\underline{v}\otimes S\underline{v})\Big)\\
&+\frac{1}{L}\Big(\alpha_\theta R_{k(\pi-2A)}\underline{v}\times(S\underline{v}\otimes S\underline{v})+\alpha_A S\underline{v}\otimes SR_{k(\pi-2A)}C\Big)+O(1)
\end{aligned}\tag{A.21}$$

where C is the conjugation matrix: $C=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Now, if we want to integrate $1/L$ in $L^p_{F(v)}(\Omega\times\mathbb{R}^2)$ for some $p>0$ we first write L in terms of x and v using the relations

$L = 2\cos A$, $|v|\cos A = x\cdot v + t|v|^2$ and the fact that t solve $|x+tv|^2 = 1$ which yield

$$L = 2\sqrt{\left(x\cdot \underline{v}\right)^2 + \left(1-|x|^2\right)}. \tag{A.22}$$

Therefore using polar change of variables

$$\begin{aligned}\iint\limits_{\Omega\times\mathbb{R}^2} \left(\frac{2}{L}\right)^p F(v)\,\mathrm{d}xv &= \iint\limits_{\Omega\times\mathbb{R}^2} \frac{1}{\left((x\cdot\underline{v})^2 + (1-|x|^2)\right)^{p/2}} F(v)\,\mathrm{d}xv \\ &= \int_0^1\int_0^{2\pi}\int_0^{2\pi} \frac{\rho_x}{\left(1-\rho_x^2\sin^2(\theta_v-\theta_x)\right)^{p/2}}\,\mathrm{d}\rho_x\,\mathrm{d}\theta_x\,\mathrm{d}\theta_v \int_{\mathbb{R}^+} F(\rho_v)\rho_v\mathrm{d}\rho_v\end{aligned}$$

where, since F is radial and normalized, $\int_{\mathbb{R}^2} F(v)\,\mathrm{d}v = 2\pi\int_{\mathbb{R}} \rho_v F(\rho_v)\mathrm{d}\rho_v = 1$. Note, in fact, that since L does not depend on the norm of $|v|$, the integrability in $L^2_{F(v)}(\Omega\times\mathbb{R}^2)$ is equivalent to the integrability in $L^2(\Omega\times\mathbb{S}^1)$ where $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$. Expanding the denominator, we have

$$\begin{aligned}&\iint\limits_{\Omega\times\mathbb{R}^2} \left(\frac{2}{L}\right)^p F(v)\,\mathrm{d}xv \qquad\qquad (\text{A.23})\\ &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\int_0^{2\pi} \frac{\rho_x}{\left(1-\rho_x|\sin(\theta_v-\theta_x)|\right)^{p/2}\left(1+\rho_x|\sin(\theta_v-\theta_x)|\right)^{p/2}}\,\mathrm{d}\rho_x\,\mathrm{d}\theta_x\,\mathrm{d}\theta_v \\ &\le C\int_0^1\int_0^{2\pi}\int_0^{2\pi} \frac{\rho_x}{\left(1-\rho_x|\sin(\theta_v-\theta_x)|\right)^{p/2}}\,\mathrm{d}\rho_x\,\mathrm{d}\theta_x\,\mathrm{d}\theta_v \\ &\le 2\pi C\int_0^1\int_0^{2\pi} \frac{\rho_x}{\left(1-\rho_x|\sin\alpha|\right)^{p/2}}\,\mathrm{d}\rho_x\,\mathrm{d}\alpha \\ &\le \tilde{C}\int_0^1\int_0^{\sqrt{1-x_2^2}} \frac{1}{(1-x_2)^{p/2}}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\le \tilde{C}\int_0^1 \frac{1}{(1-x_2)^{p/2-1/2}}\,\mathrm{d}x_2 \qquad\qquad (\text{A.24})\end{aligned}$$

hence $1/L$ will be in $L^p_{F(v)}(\Omega\times\mathbb{R}^2)$ if $p<3$.
Moreover, if we take ψ in $\mathfrak{D}_T$ (defined in beginning of this section) then we have

$$D_v^2\Big[\psi\big(\eta(x,v)\big)\Big] = D^2\eta(x,v)\nabla\psi\big(\eta(x,v)\big) + \nabla_v\eta(x,v)^T D_v^2\psi\big(\eta(x,v)\big)\nabla_v\eta(x,v).$$

The second term is uniformly bounded in x and v by (A.2) so it belongs to $L^p_{F(v)}(\Omega\times\mathbb{R}^2)$ for any $p\le\infty$. Furthermore, for the first term, we notice that for any $u\in\mathbb{R}^2$ and

$M \in \mathcal{M}_2(\mathbb{R})$ we have

$$u \times M\nabla\psi = (u \cdot \nabla\psi)M.$$

Thus the first term reads

$$\begin{aligned}
&D^2\eta(x,v)\nabla\psi\big(\eta(x,v)\big)\\
&= \frac{1}{L}\Big(\alpha_A(S\underline{v} \otimes SR_{k(\pi-2A)}C)\nabla\psi\big(\eta(x,v)\big) + \alpha_\theta\big[R_{k(\pi-2A)}\underline{v} \cdot \nabla\psi\big(\eta(x,v)\big)\big]S\underline{v} \otimes S\underline{v}\Big)\\
&\quad + \frac{1}{L}\big[SR_{k(\pi-2A)}\underline{v} \cdot \nabla\psi\big(\eta(x,v)\big)\big]\Big(\alpha_\mu^1(S\underline{v} \otimes S\underline{v}) + \alpha_\mu^2(\underline{v} \otimes S\underline{v}) + \frac{1}{L}\alpha_\mu^3(S\underline{v} \otimes S\underline{v})\Big)\\
&\quad + O(1).
\end{aligned}$$

Recall that on the boundary, $\nabla\psi(x,v) \cdot n(x) = 0$ hence, by the regularity of ψ, when $\eta(x,v)$ is close the boundary we have

$$\nabla\psi\big(\eta(x,v)\big) = \tilde{\tau}\big(\eta(x,v)\big) + O\Big(\operatorname{dist}\big(\eta(x,v),\partial\Omega\big)\Big)$$

where $\tilde{\tau}$ is the extension of the tangent $\tau(x)$ of $\partial\Omega$ at $x \in \partial\Omega$ which, since we are in the unit ball, is explicitly $\tilde{\tau}\big(\eta(x,v)\big) = \tau\big(\eta(x,v)/|\eta(x,v)|\big)$ when $|\eta(x,v)| \neq 0$. Moreover, when we start close to the grazing set the trajectory stays close to the grazing set (because A is constant close to $\pi/2$), which means $R_{k(\pi-2A)}\underline{v}$ stays close to $\tilde{\tau}(\eta(x,v))$ and in fact it will be furthest from the tangent when $\eta(x,v)$ is on the boundary where we have

$$\begin{aligned}
R_{k(\pi-2A)}\underline{v} &= \big(\cos A\big)n\big(\eta(x,v)\big) + \big(\sin A\big)\tau\big(\eta(x,v)\big)\\
&= \Big(\frac{1}{2}L\Big)n\big(\eta(x,v)\big) + \Big(1 - \frac{L^2}{4}\Big)^{1/2}\tau\big(\eta(x,v)\big)
\end{aligned}$$

so that we have

$$SR_{k(\pi-2A)}\underline{v} = n\big(\eta(x,v)\big) + O(L).$$

Finally, we can bound the distance between $\eta(x,v)$ and the boundary in terms of L because we are in a circle so the $\eta(x,v)$ is furthest from the boundary when $l_{end} = L/2$ and the Pythagorean theorem tells us in that case

$$\Big(1 - \operatorname{dist}\big(\eta(x,v),\partial\Omega\big)\Big)^2 + \Big(\frac{L}{2}\Big)^2 = 1$$

so that we have all along the trajectory

$$\mathrm{dist}\big(\eta(x,v),\partial\Omega\big) = 1 - \sqrt{1-\frac{L^2}{4}} \underset{L\ll 1}{=} \frac{L^2}{4} + o(L^2).$$

All together, these estimates yields

$$SR_{k(\pi-2A)}\underline{v}\cdot\nabla\psi\big(\eta(x,v)\big) \underset{L\ll 1}{=} O(L)$$

so that

$$\begin{aligned} D^2\eta(x,v)\nabla\psi\big(\eta(x,v)\big) \underset{L\ll 1}{=} \frac{1}{L}\Big(&\alpha_A(S\underline{v}\otimes SR_{k(\pi-2A)}C)\nabla\psi\big(\eta(x,v)\big) \\ &+\alpha_\theta\Big(R_{k(\pi-2A)}\underline{v}\cdot\nabla\psi\big(\eta(x,v)\big)\Big)S\underline{v}\otimes S\underline{v} + \alpha_\mu^3 S\underline{v}\otimes S\underline{v}\Big) + O(1). \end{aligned} \tag{A.25}$$

and from (A.24) it follows in particular that $\big\|D^2\eta(x,v)\nabla\psi\big(\eta(x,v)\big)\big\| \in L^p_{F(v)}(\Omega\times\mathbb{R}^2)$ for all $p<3$ where $\|\cdot\|$ is any matrix norm.

However that this integrability does not hold uniformly in v. Indeed, if we take the supremum over v in $\mathbb{R}^d$ of the second derivative then, close to the boundary, it behave like $1/L = 1/\sqrt{1-|x|^2}$ which is in $L^{2-\delta}(\Omega)$ for any $\delta>0$ but not in the limit when $\delta=0$, as stated in (A.4). □

A.0.3 Fractional Laplacian along the trajectories

This section of the Appendix is devoted to the proof of the following Lemma which follows from Lemma A.0.2:

Lemma III.4.2. *There exists $p>2$ such that*

$$\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,v)\big)\Big] \in L^p_{F(v)}(\Omega\times\mathbb{R}^d).$$

Proof. As we did several times before in this paper, we can split the integral formulation of the fractional Laplacian, for $R>0$, as follows

$$\begin{aligned} \big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,v)\big)\Big] = c_{d,s}P.V.\int_{|w|\le R} \frac{\psi\big(\eta(x,v)\big)-\psi\big(\eta(x,v+w)\big)}{|w|^{d+2s}}\,\mathrm{d}w \\ + c_{d,s}\int_{|w|\ge R} \frac{\psi\big(\eta(x,v)\big)-\psi\big(\eta(x,v+w)\big)}{|w|^{d+2s}}\,\mathrm{d}w \end{aligned}$$

and the integral over $|w| \geq R$ is immediately integrable in $L^p_{F(v)}(\Omega \times \mathbb{R}^d)$ for any p thanks to the boundedness of ψ in $L^\infty(\Omega)$ and the fact that F is normalized. For the integral over $w \leq R$ we do a second order Taylor-Lagrange expansion, as we did for χ_x in section III.4.2.1, in order to write for some z and $\tilde{z}$ in the ball centred at v of radius $|w|$:

$$
\begin{aligned}
&P.V. \int\limits_{|w|\leq R} \frac{\psi\big(\eta(x,v)\big) - \psi\big(\eta(x,v+w)\big)}{|w|^{d+2s}} \,\mathrm{d}w \\
&= \frac{1}{2} \int\limits_{|w|\leq R} \frac{w \cdot \Big(D^2\big[\psi(\eta(x,\cdot))\big](z) + D^2\big[\psi(\eta(x,\cdot))\big](\tilde{z})\Big) w}{|w|^{d+2s}} \,\mathrm{d}w.
\end{aligned}
$$

Let us focus on the term with z, the one with $\tilde{z}$ can obviously be handled similarly. Using (A.25) we have through straightforward computation

$$
\begin{aligned}
w \cdot D^2\Big[\psi\big(\eta(x,\cdot)\big)\Big](z) w = \frac{1}{L}\Big[&\alpha_A (w \cdot S\underline{z})\Big(\nabla\psi(\eta(x,z)) \cdot SR_{k(\pi-2A)} w\Big) \\
&+ \Big(\alpha_\theta R_{k(\pi-2A)}\underline{z} \cdot \nabla\psi\big(\eta(x,z)\big) + \alpha_\mu^3\Big)(S\underline{z} \cdot w)^2\Big] + C|w|^2
\end{aligned}
$$

where $\underline{z} = z/|z|$ and $C = C(x,z)$ is uniformly bounded in x and z. Introducing $\underline{w} = w/|w|$ as well as λ_1 and λ_2 to simplify the notations, this yields

$$
\begin{aligned}
&w \cdot D^2\Big[\psi\big(\eta(x,\cdot)\big)\Big](z) w \\
&= \frac{|w|^2}{L}\Big(\underline{w} \cdot S\underline{z}\Big)\Big(\lambda_1(S\underline{z} \cdot \underline{w}) + \lambda_2(SR_{k(\pi-2A)}\underline{w} \cdot \nabla\psi\big(\eta(x,z)\big)\Big) + C|w|^2.
\end{aligned}
$$

Therefore, using (A.22) we have

$$
\begin{aligned}
&\Bigg|\int\limits_{|w|\leq R} \frac{w \cdot D^2\big[\psi(\eta(x,\cdot))\big](z) w}{|w|^{d+2s}} \,\mathrm{d}w\Bigg| \\
&= \Bigg|\int\limits_{|w|\leq R} \frac{(\underline{w} \cdot S\underline{z})\big(\lambda_1(\underline{w} \cdot S\underline{z}) + \lambda_2\big(SR\underline{w} \cdot \nabla\psi\big(\eta(x,z)\big)\big)}{\sqrt{x \cdot \underline{z} + 1 - |x|^2}} \frac{\mathrm{d}w}{|w|^{d+2s-2}}\Bigg| \\
&\leq R^{2s} \int\limits_{\mathbb{S}^1} \frac{C_\psi}{\sqrt{x \cdot \underline{z} + 1 - |x|^2}} \mathrm{d}\underline{z}
\end{aligned}
$$

where $C_\psi = \sup_{|w|\le R}\Big((\underline{w}\cdot S\underline{z})\big(\lambda_1(\underline{w}\cdot S\underline{z}) + \lambda_2\big(SR\underline{w}\cdot\nabla\psi\big(\eta(x,z)\big)\big)\Big)$ is uniformly bounded in x and $\underline{z}$. Thus, we have for $p>0$:

$$\begin{aligned}\iint\limits_{\Omega\times\mathbb{R}^d}\Big|\big(-\Delta_v\big)^s\Big[\psi\big(t,\eta(x,v)\big)\Big]\Big|^p F(v)\,\mathrm{d}xv &\le \iint\limits_{\Omega\times\mathbb{R}^d}\Big|\int\limits_{\mathbb{S}^1}\frac{2R^{2s}C_\psi}{\sqrt{x\cdot\underline{z}+1-|x|^2}}\mathrm{d}\underline{z}\Big|^p F(v)\,\mathrm{d}xv\\ &\le \big(2R^{2s}C_\psi\big)^p\iiint\limits_{\Omega\times\mathbb{R}^d\times\mathbb{S}^1}\frac{1}{(x\cdot\underline{z}+1-|x|^2)^{p/2}}F(v)\mathrm{d}\underline{z}\,\mathrm{d}xv\end{aligned}$$

which we know to be finite if $p<3$ by (A.24) since F is radial. $\square$

A.0.4 Change of variable

Lemma III.5.4. *The change for variable F given by*

$$F\begin{pmatrix}x\\v\end{pmatrix} = \begin{pmatrix}\eta(x,v)\\-\big[\nabla_v\eta(x,v)\big]v\end{pmatrix} \tag{A.26}$$

is precisely the change of variable such that $\eta(F(x,v)) = x$ and the trajectory described by η starting at $\eta(x,v)$ with velocity $-\big[\nabla_v\eta(x,v)\big]v$ is exactly the trajectory from (x,v) backwards. Moreover, for all (x,v):

$$\det\nabla F(x,v) = 1. \tag{A.27}$$

Proof. From the explicit expression of $\nabla_v\eta(x,v)$ given above in (A.9), we see

$$-\big[\nabla_v\eta(x,v)\big]v = -R_{k(\pi-2A)}v$$

and by construction, see (A.1), we know the ending velocity of the trajectory is $R_{k(\pi-2A)}v$, see Figure A.2 for a representation, so the trajectory from $F(x,v)$ is indeed the backward trajectory from (x,v) which in particular implies that $\eta(F(x,v)) = x$. In order to compute the determinant of F we need the Jacobian with respect to x of η. Following the same line of arguments as for the Jacobian in v we write

$$\begin{aligned}\nabla_x\eta(x,v) = &\Big[SR_{k(\pi-2A)}\big(x+v-k(z_0-z_{-1})\big)\Big]\otimes\big(-2k\nabla_x A\big)\\ &-kR_{k(\pi-2A)}\nabla_x\big(z_0-z_{-1}\big)+R_{k(\pi-2A)}.\end{aligned}$$

where, from the relations we used to derive (A.5) and (A.6) we have

$$\nabla_x \theta = \frac{2}{L} S \frac{v}{|v|}, \qquad \nabla_x A = \frac{-2}{L} S \frac{v}{|v|} \tag{A.28}$$

which yields

$$\nabla_x z_0 = \frac{2}{L} S z_0 \otimes S \frac{v}{|v|}, \qquad \nabla_x z_{-1} = \frac{-2}{L} S z_{-1} \otimes S \frac{v}{|v|}. \tag{A.29}$$

As a consequence

$$\nabla_x \eta(x,v) = S R_{k(\pi-2A)} \left[\frac{4k}{L} \big(v - k(z_0 - z_{-1}) \big) + \frac{2k}{L} \big(2s - z_0 - z_{-1} \big) \right] \otimes S \frac{v}{|v|} + R_{k(\pi-2A)}$$

and using (A.8) we get

$$\nabla_x \eta(x,v) = 2k \left(2 \frac{l_{end}}{L} - 1 \right) S R_{k(\pi-2A)} \frac{v}{|v|} \otimes S \frac{v}{|v|} + R_{k(\pi-2A)}. \tag{A.30}$$

We also need the Jacobian matrices of $-\big[\nabla_v \eta(x,v)\big] v$ which are

$$\nabla_x \Big(-\big[\nabla_v \eta(x,v)\big] v \Big) = \frac{-4k|v|}{L} S R_{k(\pi-2A)} \frac{v}{|v|} \otimes S \frac{v}{|v|}, \tag{A.31}$$

$$\nabla_v \Big(-\big[\nabla_v \eta(x,v)\big] v \Big) = 2k \left(1 - 2 \frac{l_{in}}{L} \right) S R_{k(\pi-2A)} \frac{v}{|v|} \otimes S \frac{v}{|v|} - R_{k(\pi-2A)}. \tag{A.32}$$

With appropriate coefficient α_x, α_v, β_x, β_v (which are functions of x and v), using the angular function Θ defined in (A.10) and writing R instead of $R_{k(\pi-2A)}$ we can then write the Jacobian of F as the following sum of block matrices

$$\begin{aligned} \nabla F(x,v) &= \begin{pmatrix} \nabla_x \eta(x,v) & \nabla_v \eta(x,v) \\ \nabla_x \Big(-\big[\nabla_v \eta(x,v)\big] v \Big) & \nabla_v \Big(-\big[\nabla_v \eta(x,v)\big] v \Big) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_x \Theta & \alpha_v \Theta \\ \beta_x \Theta & \beta_v \Theta \end{pmatrix} + \begin{pmatrix} R & R \\ 0 & -R \end{pmatrix}. \end{aligned}$$

Now, notice that $R^{-1}\Theta = S\frac{v}{|v|} \otimes S\frac{v}{|v|} := N$ which yields the relation

$$\det\left(\begin{pmatrix} R^{-1} & R^{-1} \\ 0 & -R^{-1} \end{pmatrix} \nabla F(x,v) \right) = \det\left(\begin{pmatrix} (\alpha_x + \beta_x) N & (\alpha_v + \beta_v) N \\ -\beta_x N & -\beta_v N \end{pmatrix} + \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \right)$$

where we also notice that

$$\det\begin{pmatrix} R^{-1} & R^{-1} \\ 0 & -R^{-1} \end{pmatrix} = \det\Big(-R^{-2}\Big) = 1$$

because it is a rotation matrix in dimension 2. Therefore,

$$\det \nabla F(x,v) = \det\begin{pmatrix} (\alpha_x+\beta_x)N + Id & (\alpha_v+\beta_v)N \\ -\beta_x N & -\beta_v N + Id \end{pmatrix}.$$

Finally, it is rather simple to find the eigenvalues of this matrix. Indeed, since $Nv = (v\cdot S\frac{v}{|v|})S\frac{v}{|v|} = 0$ we see that the 4-dimensional vectors $(v,0)$ and $(0,v)$ are both eigenvectors associated with the eigenvalue 1. Moreover, we notice that $NSv = Sv$ so we solve for λ and μ the equation

$$\begin{pmatrix} (\alpha_x+\beta_x)N + Id & (\alpha_v+\beta_v)N \\ -\beta_x N & -\beta_v N + Id \end{pmatrix}\begin{pmatrix} Sv \\ \lambda Sv \end{pmatrix} = \mu\begin{pmatrix} Sv \\ \lambda Sv \end{pmatrix}$$

and find the two remaining eigenvalues:

$$\mu_1 = 1 - 2k\big(k+\sqrt{k^2-1}\big)$$
$$\mu_2 = 1 - 2k\big(k-\sqrt{k^2-1}\big).$$

Note that in order to find those values we used the relations $\alpha_x+\beta_x-\beta_v = -4k^2$ and $\beta_v\alpha_x - \beta_x\alpha_v = -4k^2$ which are deduced easily from the expressions (A.9) (A.30) (A.31) and (A.32). In the end, we get the determinant of $\nabla F(x,v)$:

$$\det\nabla F(x,v) = \Big(1-2k\big(k+\sqrt{k^2-1}\big)\Big)\Big(1-2k\big(k-\sqrt{k^2-1}\big)\Big) = 1.$$

□

A.0.5 Control of the Laplacian of η

The purpose of this section is to prove Lemma IV.5.1 from Chapter IV which reads

Lemma IV.5.1. *For all $\psi \in \mathfrak{D}_T$ we have*

$$\sup_{r>0}\Big(\Delta_v\left[\psi\left(t,\eta(x,\cdot)\right)\right](rv)\Big) \in \mathrm{L}^\infty\big((0,T);\mathrm{L}^2(\Omega\times\mathbb{S}^{d-1})\big) \qquad \text{(IV.36)}$$

Proof. First, let us recall that

$$\begin{aligned}\Delta_v\left[\psi\left(t,\eta(x,v)\right)\right] &= \Delta_v\eta(x,v)\cdot\nabla_x\psi\left(t,\eta(x,v)\right)\\ &\quad + \mathrm{Tr}\left(\nabla_v\eta(x,v)^{\top}\nabla_v\eta(x,v)H_x\psi\left(t,\eta(x,v)\right)\right).\end{aligned}$$

From the previous sections of this appendix, we recall that $\nabla_v\eta(x,v)$ is uniformly bounded in x and v, so the second term in the above expression is immediately handled. For the first term, from the previous expression of $D^2\eta(x,v)$, it is easy to see that the Laplacian of η can be written as

$$\Delta\eta(x,v) = \frac{1}{L^2}\lambda S R_{k(\pi-2A)}\frac{v}{|v|} + C$$

where $\lambda = \lambda(x,v)$ and $C = C(x,v)$, both uniformly bounded in x and v, S is the symmetry matrix: $S = \Big(0 \quad 1; -1, 0\Big)$, and $R_{k(\pi-2A)}$ is the rotation matrix of angle $k(\pi-2A)$.
Moreover, when we start close to the grazing set, the trajectory stays close to the grazing set (because A is a constant close to $\pi/2$), which means $R_{k(\pi-2A)}v/|v|$ stays close to $\tau(\eta(x,v))$, then tangent of Ω at $\eta(x,v)/|\eta(x,v)| \in \partial\Omega$. In fact it will be furthest from the tangent when $\eta(x,v)$ is on the boundary where we have

$$\begin{aligned}R_{k(\pi-2A)}\frac{v}{|v|} &= \big(\cos A\big)n\big(\eta(x,v)\big) + \big(\sin A\big)\tau\big(\eta(x,v)\big)\\ &= \Big(\frac{1}{2}L\Big)n\big(\eta(x,v)\big) + \Big(1-\frac{L^2}{4}\Big)^{1/2}\tau\big(\eta(x,v)\big)\end{aligned}$$

so that

$$S R_{k(\pi-2A)}\frac{v}{|v|} = n\big(\eta(x,v)\big) + O(L)$$

where $n(\eta(x,v))$ is the outward normal at $\eta(x,v)/|\eta(x,v)| \in \partial\Omega$.
Furthermore, if we consider $\psi \in \mathfrak{D}_T$ then on the boundary, $\nabla\psi(x,v)\cdot n(x) = 0$ hence, by the regularity of ψ, when $\eta(x,v)$ is close the boundary we have

$$\nabla\psi\big(\eta(x,v)\big)\Big) = \tau\big(\eta(x,v)\big) + O\Big(\mathrm{dist}\big(\eta(x,v),\partial\Omega\big)\Big).$$

We can bound the distance between $\eta(x,v)$ and the boundary in terms of L because we are in a circle so the $\eta(x,v)$ is furthest from the boundary when it is in the middle between two reflections and the Pythagorean theorem tells us in that case

$$\Big(1 - \operatorname{dist}\big(\eta(x,v), \partial\Omega\big)\Big)^2 + \Big(\frac{L}{2}\Big)^2 = 1$$

so that we have all along the trajectory

$$\operatorname{dist}\big(\eta(x,v), \partial\Omega\big) = 1 - \sqrt{1 - \frac{L^2}{4}} = \frac{L^2}{4} + o(L^2).$$

All together, this yields

$$\begin{aligned}\Delta\eta(x,v) \cdot \nabla\psi\big(\eta(x,v)\big) &= \frac{\lambda}{L} S R_{k(\pi-2A)} \frac{v}{|v|} \cdot \nabla\psi\big(\eta(x,v)\big) + O(1) \\ &= O\Big(\frac{1}{L}\Big).\end{aligned}$$

The integrability of $1/L$ that we established at the end of Section A.0.2 concludes the proof. Note, as a remark, that the bound is not uniform in v, as we explained in Section A.0.2, which is why the bound we write is only homogeneous with respect to the norm $|v|$, and if we took the supremum in v instead then $1/L$ would be equivalent to $1/\sqrt{1-|x|^2}$ which is in $L^{2-\delta}(\Omega)$ for all $\delta > 0$ but not for $\delta = 0$. □

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